3–3 CONTOUR INTEGRATION

One of the most powerful means for evaluating definite integrals is provided by the theorem of residues from the theory of functions of a complex variable. We shall illustrate this method of contour integration by a number of examples in this section. Before reading this material, the student who does not know the theory of functions of a complex variable reasonably well should review (or learn) certain parts of this theory. These parts are presented in the Appendix of this book to serve as an aid in the review (or as a guide to the study).

The theorem of residues [Appendix, Eq. (A-15)] tells us that if a function $f(z)$ is regular in the region bounded by a closed path C, except for a finite number of poles and isolated essential singularities in the interior of C, then the integral of $f(z)$ along the contour C is

$$\int_C f(z)\,dz = 2\pi i \sum \text{residues}$$

where $\sum$ residues means the sum of the residues at all the poles and essential singularities inside C.

The residues at poles and isolated essential singularities may be found as follows.

If $f(z)$ has a simple pole (pole or order one) at $z = z_0$, the residue is

$$a_{-1} = [(z - z_0)f(z)]_{z=z_0} \tag{3-32}$$

If $f(z)$ is written in the form $f(z) = q(z)/p(z)$, where $q(z)$ is regular and $p(z)$ has a simple zero at z_0, the residue of $f(z)$ at z_0 may be computed from

$$a_{-1} = \frac{q}{p'}\bigg|_{z=z_0} \tag{3-33}$$

If z_0 is a pole of order n, the residue is

$$a_{-1} = \frac{1}{(n-1)!}\left\{\left(\frac{d}{dz}\right)^{n-1}[(z - z_0)^n f(z)]\right\}_{z=z_0} \tag{3-34}$$

If z_0 is an isolated essential singularity, the residue is found from the Laurent expansion (Appendix, Section A–2, item 7).

We illustrate the method of contour integration by some examples.

EXAMPLE

$$I = \int_0^\infty \frac{dx}{1 + x^2} \tag{3-35}$$

Consider $\oint dz/(1 + z^2)$ along the contour of Figure 3–1. Along the real axis the integral is $2I$. Along the large semicircle in the upper-half plane we get zero, since

$$z = Re^{i\theta} \qquad dz = iRe^{i\theta}\,d\theta \qquad \frac{1}{1 + z^2} \approx \frac{e^{-2i\theta}}{R^2}$$

$$\int \frac{dz}{1 + z^2} \approx \frac{i}{R}\int e^{-i\theta}\,d\theta \to 0 \text{ as } R \to \infty$$

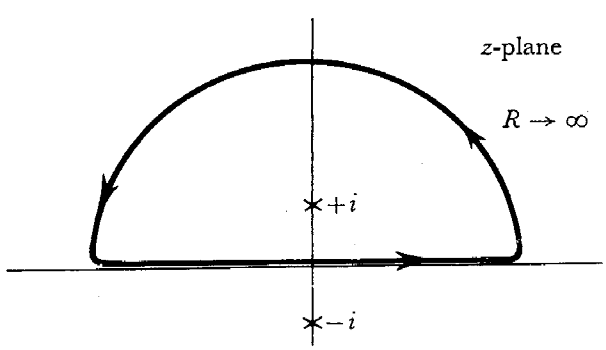

Figure 3–1 Contour for the integral (3-35)

The residue of $1/(1+z^2) = 1/(z+i)(z-i)$ at $z = i$ is $1/(2i)$. Thus

$$2I = 2\pi i\left(\frac{1}{2i}\right) = \pi \qquad I = \frac{\pi}{2}$$

Note that an important part of the problem may be choosing the "return path" so that the contribution from it is simple (preferably zero).

EXAMPLE

Consider a resistance R and inductance L connected in series with a voltage $V(t)$ (Figure 3–2). Suppose $V(t)$ is a voltage impulse, that is, a very high pulse lasting for a very short time. As we shall see in Chapter 4, we can write to a good approximation

$$V(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$$

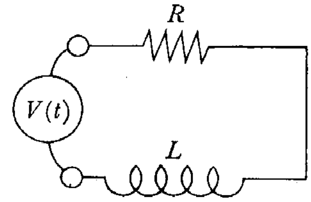

Figure 3–2 Series *R-L* circuit

where A is the area under the curve $V(t)$.

The current due to a voltage $e^{i\omega t}$ is $e^{i\omega t}/(R + i\omega L)$. Thus the current due to our voltage pulse is

$$I(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{R + i\omega L} \tag{3-36}$$

Let us evaluate this integral.

If $t < 0$, the integrand is exponentially small for $\text{Im}\,\omega \to -\infty$, so that we may complete the contour by a large semicircle in the *lower*-half ω-plane, along which the integral vanishes.[3] The contour encloses no singularities, so that $I(t) = 0$.

If $t > 0$, we must complete the contour by a large semicircle in the *upper*-half plane. Then

$$I(t) = 2\pi i\left(\frac{A}{2\pi}\right)\frac{e^{-Rt/L}}{iL} = \frac{A}{L}e^{-Rt/L}$$

[3] A rigorous justification of this procedure is provided by *Jordan's lemma*; see Copson (C8) p. 137 for example.