1 Notes on Fourier series of periodic functions

1.1 Background

Any temporal function can be represented by a multiplicity of basis sets. When the function is assumed to exist for all of time, a not unreasonable approximation for real signals in the steady state, the optimal representation is in the frequency domain. Here we express function $V(t)$ in terms of a continuous expansion in sines and cosines, which are most conveniently written in their complex forms, *i.e.*, $\sin x = (e^{ix} - e^{-ix})/(2i)$ and $\cos x = (e^{ix} + e^{-ix})/2$. Then

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \tilde{V}(\omega) \; e^{i\omega t} \tag{1.1}$$

where $\tilde{V}(\omega)$ is a complex function that sets the contribution of different frequencies, ω. The inverse transform is

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega t}. \tag{1.2}$$

When $V(t)$ is periodic with period T, so that $V(t+T) = V(t)$, we have

$$\int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega t} \;=\; \int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega(t+T)} \tag{1.3}$$

so that

$$e^{-i\omega T} \;=\; 1 \tag{1.4}$$

and ω can only take on discrete values, *i.e.*,

$$\omega \;=\; 0, \; \pm\frac{2\pi}{T}, \; \pm\frac{4\pi}{T}, \; \pm\frac{6\pi}{T}, \; \ldots\,. \tag{1.5}$$

We define $\omega_o = 2\pi/T$ so that

$$\omega \;=\; 0, \; \pm\omega_o, \; \pm 2\omega_o, \; \pm 3\omega_o, \; \ldots\,. \tag{1.6}$$

and write the Fourier expansion in a discrete form, *i.e.*,

$$V(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \; e^{ik\omega_o t}. \tag{1.7}$$

The $\tilde{c}_k$ are complex numbers that weight each of the harmonics ω_o. They are defined by

$$\tilde{c}_k = \frac{1}{T} \int_{-T/2}^{+T/2} dt \; V(t) \; e^{-ik\omega_o t}. \tag{1.8}$$

Since we need to end up with sines and cosines, the constant are constrained so that $\tilde{c}_k = \tilde{c}_{-k}$ for $\tilde{c}_k$ real and $\tilde{c}_k = -\tilde{c}_{-k}$ for $\tilde{c}_k$ imaginary.

1.2 Sine waves

This is a trivial case. We have:

$$\begin{aligned} \tilde{c}_0 &= 0 \\ \tilde{c}_{\pm 1} &= \pm \frac{1}{2i} \\ \tilde{c}_{\pm k;\, k \geq 2} &= 0 \end{aligned} \tag{1.9}$$

1.3 Square waves

Here $V(-T/2 < t < 0) =$ -1 and $V(0 < t < T/2) = +1$. We have:

$$\begin{aligned} \tilde{c}_k &= \frac{1}{T} \left[\int_{-T/2}^{0} dt \; (-1) \; e^{-ik\omega_o t} \; + \; \int_{0}^{+T/2} dt \; (+1) \; e^{-ik\omega_o t} \right] \\ &= \frac{1}{-ik\omega_o T} \left[- \; e^{-ik\omega_o t}|_{-T/2}^{0} \; + \; e^{-ik\omega_o t}|_{0}^{T/2} \right] \\ &= \frac{1}{-ik\omega_o T} \left[-1 \; + \; e^{+ik\omega_o T/2} \; + \; e^{-ik\omega_o T/2} \; - \; 1 \right] \\ &= \frac{1}{ik\omega_o T} \left[2 \; - 2 \; cos \left(k\omega_o T/2 \right) \right]. \end{aligned} \tag{1.10}$$

We recall that $\omega_o T = 2\pi$, so that

$$\begin{aligned} \tilde{c}_k &= \left(\frac{4}{\pi}\right) \left(\frac{1}{2i}\right) \left[\frac{1 \; - \; cos\left(\pi k\right)}{2k} \right] \\ &= \left(\frac{4}{\pi}\right) \left(\frac{1}{2i}\right) \left(..., \; 0, \; -\frac{1}{5}, \; 0, \; -\frac{1}{3}, \; 0, \; -1, \; 0, \; +1, \; 0, \; +\frac{1}{3}, \; 0, \; +\frac{1}{5}, \; 0, \; ... \right) \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} V(t) &= \frac{4}{\pi} \left(\frac{... - \frac{1}{5} e^{-i5\omega_o t} - \frac{1}{3} e^{-i3\omega_o t} - e^{-i\omega_o t} + e^{i\omega_o t} + \frac{1}{3} e^{i3\omega_o t} + \frac{1}{5} e^{i5\omega_o t} + ...}{2i} \right) \\ &= \frac{4}{\pi} \left[sin(\omega_o t) \; + \; \frac{1}{3} sin(3\omega_o t) \; + \; \frac{1}{5} sin(5\omega_o t) \; + \; ... \right]. \end{aligned} \tag{1.12}$$

The result is that the square wave is constructed from a weakly converging set of odd harmonics.
Of potential interest, the "smoother" the function the faster the series for $\tilde{c}_k$ converges, *i.e.*, $\tilde{c}_k$ = constant for a period series of delta functions, *i.e.*, the so-called comb function, $\tilde{c}_k \propto 1/k$ for a square wave as derived above, $\tilde{c}_k \propto 1/k^2$ for a triangular wave, *etc.*