1 Notes on Fourier series of periodic functions

1.1 Background

Any temporal function can be represented by a multiplicity of basis sets. When the function is assumed to exist for all of time, a not unreasonable approximation for real signals in the steady state, the optimal representation is in the frequency domain. Here we express function \( V(t) \) in terms of a continuous expansion in sines and cosines, which are most conveniently written in their complex forms, \( i.e. \), \( \sin x = (e^{ix} - e^{-ix})/(2i) \) and \( \cos x = (e^{ix} + e^{-ix})/2 \). Then

\[
V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \tilde{V}(\omega) \, e^{i\omega t}
\]

(1.1)

where \( \tilde{V}(\omega) \) is a complex function that sets the contribution of different frequencies, \( \omega \). The inverse transform is

\[
\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt \, V(t) \, e^{-i\omega t}.
\]

(1.2)

When \( V(t) \) is periodic with period \( T \), so that \( V(t + T) = V(t) \), we have

\[
\int_{-\infty}^{\infty} dt \, V(t) \, e^{-i\omega t} = \int_{-\infty}^{\infty} dt \, V(t) \, e^{-i\omega(t+T)}
\]

(1.3)

so that

\[
e^{-i\omega T} = 1
\]

(1.4)

and \( \omega \) can only take on discrete values, \( i.e., \)

\[
\omega = 0, \pm \frac{2\pi}{T}, \pm \frac{4\pi}{T}, \pm \frac{6\pi}{T}, \ldots
\]

(1.5)

We define \( \omega_o = 2\pi/T \) so that

\[
\omega = 0, \pm \omega_o, \pm 2\omega_o, \pm 3\omega_o, \ldots
\]

(1.6)

and write the Fourier expansion in a discrete form, \( i.e., \)

\[
V(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \, e^{ik\omega_o t}.
\]

(1.7)

The \( \tilde{c}_k \) are complex numbers that weight each of the harmonics \( \omega_o \). They are defined by
\[ \tilde{c}_k = \frac{1}{T} \int_{-T/2}^{+T/2} dt \ V(t) \ e^{-i\omega_0 t}. \]  

(1.8)

Since we need to end up with sines and cosines, the constant are constrained so that \( \tilde{c}_k = \tilde{c}_{-k} \) for \( \tilde{c}_k \) real and \( \tilde{c}_k = -\tilde{c}_{-k} \) for \( \tilde{c}_k \) imaginary.

1.2 Sine waves

This is a trivial case. We have:

\[ \tilde{c}_0 = 0 \]  

(1.9)

\[ \tilde{c}_{\pm 1} = \pm \frac{1}{2i} \]

\[ \tilde{c}_{\pm k; k \geq 2} = 0 \]

1.3 Square waves

Here \( V(-T/2 < t < 0) = -1 \) and \( V(0 < t < T/2) = +1 \). We have:

\[ \tilde{c}_k = \frac{1}{T} \left[ \int_{-T/2}^{0} dt \ (1) e^{-i\omega_0 t} + \int_{0}^{+T/2} dt \ (1) e^{-i\omega_0 t} \right] \]  

(1.10)

\[ = \frac{1}{-i\omega_0 T} \left[ -e^{-i\omega_0 t}|_{T/2} - e^{-i\omega_0 t}|_{0} \right] \]

\[ = \frac{1}{-i\omega_0 T} \left[ -1 + e^{i\omega_0 T/2} + e^{-i\omega_0 T/2} - 1 \right] \]

\[ = \frac{1}{i\omega_0 T} \left[ 2 - 2 \cos (k\omega_0 T/2) \right]. \]

We recall that \( \omega_0 T = 2\pi \), so that

\[ \tilde{c}_k = \left( \frac{4}{\pi} \right) \left( \frac{1}{2i} \right) \left[ \frac{1 - \cos (\pi k)}{2k} \right] \]  

(1.11)

\[ = \left( \frac{4}{\pi} \right) \left( \frac{1}{2i} \right) \left( ..., 0, -\frac{1}{5}, 0, -\frac{1}{3}, 0, -1, 0, +1, 0, +\frac{1}{3}, 0, +\frac{1}{5}, 0, ... \right) \]

and

\[ V(t) = \frac{4}{\pi} \left( \frac{\sin(\omega_0 t) + 1/3 \sin(3\omega_0 t) + 1/5 \sin(5\omega_0 t) + ...}{2i} \right) \]  

(1.12)
The result is that the square wave is constructed from a weakly converging set of odd harmonics. Of potential interest, the "smoother" the function the faster the series for $\tilde{c}_k$ converges, \textit{i.e.}, $\tilde{c}_k$ = constant for a period series of delta functions, \textit{i.e.}, the so-called comb function, $\tilde{c}_k \propto 1/k$ for a square wave as derived above, $\tilde{c}_k \propto 1/k^2$ for a triangular wave, \textit{etc.}