6.3 FOURIER SERIES

The exponential Fourier series for a periodic signal was developed in Section 5.4 as a linear combination of orthonormal functions having the property of being a least-square-error approximation. Also shown was the fact that when the signal in question has well-defined average power, its Fourier series may be deemed a valid signal representation for most engineering purposes. That premise is adopted here, where we put the theory to work. Specifically, we shall use the Fourier series to express periodic signals as sums of phasors, from which their line spectra drop out immediately. Then, in Section 6.5, the phasor sum is employed to carry out periodic steady-state analysis.

Exponential Fourier series

Let $v(t)$ be a periodic power signal with fundamental period T_0. It can be expanded as a linear combination of phasors via the exponential Fourier series

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega t} \tag{1}$$

where Ω is the fundamental angular frequency

$$\Omega = \frac{2\pi}{T_0} \tag{2}$$

†It also happens in this case that the corner frequency numerically equals the half-power bandwidth and, at that point, $|H(B)|_{\text{db}} = -20 \log_{10} (1/\sqrt{2}) = 3.01 \approx -3$ db, which explains why B commonly is referred to as the *3-db bandwidth*.

and the coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{T_0} v(t) e^{-jn\Omega t} dt \tag{3}$$

As in Chapter 5, the symbol $\int_{T_0}$ stands for integration over any period $t_1 \leq t \leq t_1 + T_0$, with t_1 being arbitrary.

Perhaps the most striking feature of the series is that only the c_n's depend on the explicit details of the signal's behavior, all else being predetermined once it is known that $v(t)$ is periodic. Putting this another way, the Fourier-series expansions for all power signals having the same period differ only in the coefficients. For this reason we concentrate our discussion on the properties of c_n implied by (3), especially the following four points.

First, the c_n's are generally *complex* quantities, whether or not the signal is complex. To illustrate, if $v(t)$ is in fact real (noncomplex), then the real and imaginary parts of c_n are given by

$$c_n = \underbrace{\left[\frac{1}{T_0} \int_{T_0} v(t) \cos n\Omega t dt\right]}_{\text{Re } [c_n]} + j \underbrace{\left[-\frac{1}{T_0} \int_{T_0} v(t) \sin n\Omega t dt\right]}_{\text{Im } [c_n]} \tag{4}$$

which comes about from applying Euler's law to $e^{-jn\Omega t}$.

Second, because (3) is a definite integral with t the variable of integration, c_n is *independent of time*. Underscoring this fact, we introduce the change of variable $\psi = \Omega t$ so that (3) becomes

$$c_n = \frac{1}{2\pi} \int_{2\pi} v\left(\frac{\psi}{\Omega}\right) e^{-jn\psi} d\psi \tag{5}$$

which, with its compact form, is sometimes handy when calculating c_n or demonstrating other relationships.

Third, setting $n = 0$ in (3), the zeroth-order coefficient c_0 is

$$c_0 = \frac{1}{T_0} \int_{T_0} v(t) dt \tag{6}$$

which, upon examination, should be recognized as the *time-average value* of $v(t)$. Thus, the constant term in the Fourier series is just the average value of the signal.

Fourth, if the signal in question is a *real* function of time, then the

negative and positive coefficients are related simply by *complex conjugation*, namely

$$c_{-n} = c_n^* \tag{7}$$

as follows from (4) by replacing n with $-n$.

Waveform symmetry

Besides these basic properties, there are certain simplifications when $v(t)$ has *symmetry* of one type or another. In particular, if $v(t)$ is an *even* function then the integrand in the first term of (4) has even symmetry while the second has odd symmetry. Therefore, taking the range of integration to be $-T_0/2 \leq t \leq T_0/2$ and invoking Eq. (20), Sect. 5.1,

$$c_n = \frac{2}{T_0}\int_0^{T_0/2} v(t)\cos n\Omega t\, dt \tag{8a}$$

By the same procedure, if $v(t)$ is an *odd* function,

$$c_n = -j\frac{2}{T_0}\int_0^{T_0/2} v(t)\sin n\Omega t\, dt \tag{8b}$$

and it follows that $c_0 = 0$.

Another type of symmetry, called *half wave* or *rotation* symmetry, is defined by the property

$$v\left(t \pm \frac{T_0}{2}\right) = -v(t) \tag{9}$$

which can hold only for a periodic signal. As illustrated in Figure 6.7, such signals retain odd symmetry even when the time origin is shifted an integer number of half periods in either direction. Under this condition it

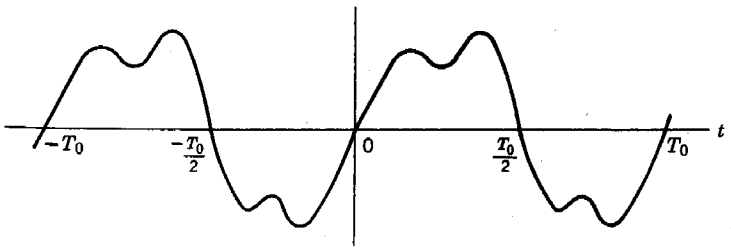

Figure 6.7. An example of half-wave symmetry.

turns out that

$$c_n = 0 \qquad n = 0, \pm 2, \pm 4, \ldots \tag{10}$$

so the Fourier series consists entirely of odd-order terms.

Occasionally one encounters signals with *hidden symmetry*, i.e., actual symmetry that is obscured by the presence of an additive constant term, or potential symmetry that can be realized by shifting the location of the time origin. In the former case the constant can be subtracted out and then added to the zeroth coefficient c_0 after performing the Fourier-series expansion of the symmetric signal. In the latter case, since the time origin is not physically unique we are usually free to redefine it so as to gain simplifications of symmetry. A theorem that covers the effects of time shifting will be presented in Section 6.4

Further understanding of waveform symmetry is gained by examining the *trigonometric* Fourier series.

Trigonometric Fourier series

When $v(t)$ is a *real* signal – and hence $c_{-n} = c_n^*$ – its exponential Fourier series can be converted to a trigonometric form by the following manipulation. Regrouping pairwise all but the zeroth term in (1), so the summation index is always positive, we have

$$v(t) = c_0 + \sum_{n=1}^{\infty} (c_n e^{jn\Omega t} + c_{-n} e^{-jn\Omega t})$$

But $c_n e^{jn\Omega t}$ and $c_{-n} e^{-jn\Omega t}$ now form a complex-conjugate pair so, with c_n in polar form,

$$v(t) = c_0 + \sum_{n=1}^{\infty} |2c_n| \cos (n\Omega t + \arg [c_n]) \tag{11}$$

which expresses $v(t)$ as a sum of sinusoidal waves with various amplitudes and phase angles, and all terms in (11) are real. Another trigonometric form, involving both sines and cosines, can be derived, but (11) is generally more useful in systems analysis than the sine-cosine series.

Now if a real signal has *even symmetry* then, according to (8a), the series coefficients c_n are strictly real. Therefore, the trigonometric series involves only terms of the form $\pm|2c_n| \cos n\Omega t$, where the minus sign is needed when c_n is negative. Since the signal is even, it is only natural that the series should reduce to a sum of *even functions*, $\cos n\Omega t$. Similarly,

when $v(t)$ has *odd symmetry* the series terms become $\pm|2c_n|\sin n\Omega t$, so we have a sum of *odd functions*, $\sin n\Omega t$.

These comments, along with our previous observations, are best illustrated by a few examples of calculating the Fourier series of a given signal.

Example 6.2 A rectangular pulse train

The rectangular pulse train of Figure 6.8 is a very important idealized signal. Formally, it is written by specifying its value over one period and citing the periodicity requirement, as below:

$$v(t) = \begin{cases} A & |t| \leq \tau/2 \\ 0 & \tau/2 < |t| \leq T_0/2 \end{cases}$$

$$v(t) = v(t \pm mT_0) \qquad m = 0, \pm 1, \pm 2, \ldots$$

The parameter τ is the pulse *duration* and A the *amplitude*.

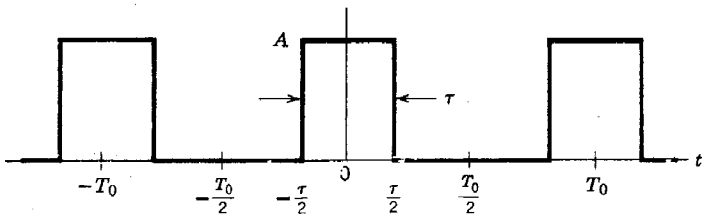

Figure 6.8. Rectangular pulse train.

Since the integration for c_n is straightforward, we shall ignore the fact that $v(t)$ has even symmetry and use the basic expression (3) to find the series coefficients. Taking the integration limits as $-T_0/2$ and $T_0/2$ gives

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} v(t)e^{-jn\Omega t}dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} Ae^{-jn\Omega t}dt$$

$$= \frac{A}{-jn\Omega T_0}(e^{-jn\Omega\tau/2} - e^{jn\Omega\tau/2})$$

$$= \frac{A}{\pi n}\sin\frac{\pi n\tau}{T_0}$$

where we have used $e^{j\phi} - e^{-j\phi} = 2j\sin\phi$ and substituted $\Omega = 2\pi/T_0$.

This expression can be further tidied up by introducing a new function, called the *sinc function*,† defined as

$$\operatorname{sinc} z \triangleq \frac{\sin \pi z}{\pi z} \tag{12}$$

and plotted in Figure 6.9. Being the product of two odd functions, sinc z is an *even* function and, with sin πz in the numerator, it has *zero crossings* at all nonzero integer values of its argument, i.e.,

$$\operatorname{sinc} z = 0 \qquad z = \pm 1, \pm 2, \ldots \tag{13a}$$

while the indeterminate case of $z = 0$ yields

$$\operatorname{sinc} 0 = \lim_{z \to 0} \frac{\sin \pi z}{\pi z} = 1 \tag{13b}$$

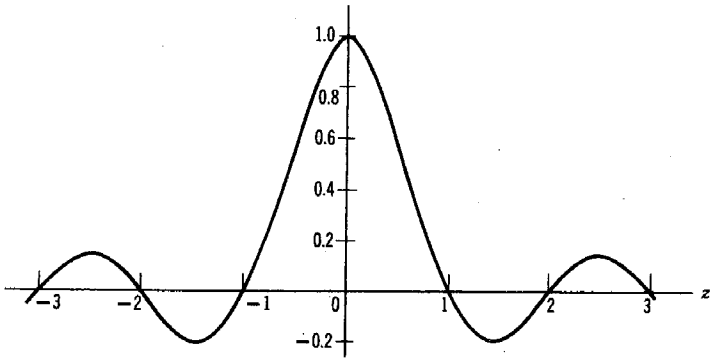

Figure 6.9. The sinc function $\operatorname{sinc} z \triangleq \frac{\sin \pi z}{\pi z}$.

Table 6.1

z	sinc z	z	sinc z	z	sinc z
0.0	1.000	1.0	0.000	2.5	0.127
0.2	0.935	1.2	−0.156	3.5	−0.091
0.4	0.757	1.4	−0.216	4.5	0.071
0.6	0.505	1.6	−0.189	5.5	−0.058
0.8	0.234	1.8	−0.104	6.5	0.049

†Some authors define instead $Sa(z) \triangleq (\sin z)/z$, called the *sampling function* because of another context in which it occurs. The two functions differ only by a factor of π in the argument, i.e., $\operatorname{sinc} z = Sa(\pi z)$.

as found by simple limiting. Selected values of sinc z are listed in Table 6.1.

Using the sinc-function notation, the series coefficients become

$$c_n = \frac{A\tau}{T_0} \frac{\sin(\pi n\tau/T_0)}{(\pi n\tau/T_0)}$$
$$= \frac{A\tau}{T_0} \operatorname{sinc} \frac{n\tau}{T_0} \tag{14}$$

which we note is strictly real – because $v(t)$ is real and even – and independent of time. Setting $n = 0$ yields

$$c_0 = \frac{A\tau}{T_0}$$

which clearly is the average value of the signal. Finally, inserting (14) into (1) gives

$$v(t) = \frac{A\tau}{T_0} \sum_{n=-\infty}^{\infty} \operatorname{sinc} \frac{n\tau}{T_0} e^{jn\Omega t} \tag{15a}$$

for the exponential Fourier series representation of the rectangular pulse train. Or, since $v(t)$ is real and even, its trigonometric Fourier series can be written as

$$v(t) = \frac{A\tau}{T_0}\left(1 + \sum_{n=1}^{\infty} 2 \operatorname{sinc} \frac{n\tau}{T_0} \cos n\Omega t\right) \tag{15b}$$

Example 6.3 A half-rectified sine wave

Passing a sine wave of angular frequency Ω through a half-wave rectifier produces the signal shown in Figure 6.10. There is no symmetry here,

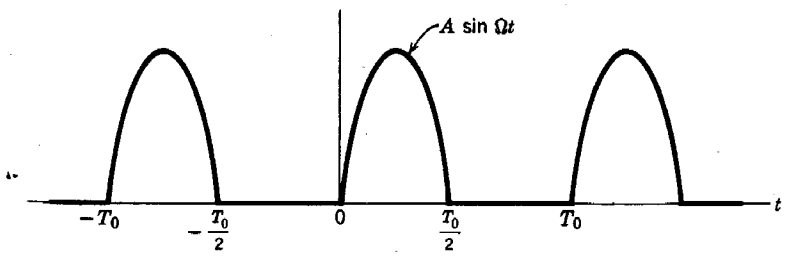

Figure 6.10. Half-rectified sine wave.

but c_n can be found using (5) and a table of integrals; i.e., $T_0 = 2\pi/\Omega$, $v(\psi/\Omega) = A \sin \psi$ for $0 \leq \Omega t \leq \pi$, and

$$c_n = \frac{1}{2\pi} \int_0^{\pi} A \sin \psi \, e^{-jn\psi} d\psi$$

$$= \frac{A}{2\pi(1-n^2)} (1 + e^{-jn\pi}) \qquad n^2 \neq 1$$

But $(1 + e^{-jn\pi})$ equals $+2$ when n is even and 0 when n is odd, so

$$c_n = \begin{cases} \dfrac{A}{\pi(1-n^2)} & n = 0, \pm 2, \pm 4, \ldots \\ 0 & n = \pm 3, \pm 5, \ldots \end{cases} \tag{16a}$$

and all odd-order terms are zero save for the indeterminate case of $n = \pm 1$. By separate integration or limiting one finds for that case

$$c_{\pm 1} = \mp j \frac{A}{4} \tag{16b}$$

Example 6.4 Sinusoidal waves

Both previous examples have resulted in series representations with an *infinite* number of terms. This is usually the case, but not always, an important counter example being the sinusoidal signal

$$v(t) = A \cos(\Omega t + \theta)$$

Without bothering with integration, the series coefficients can be found directly from the phasor representation

$$v(t) = \underbrace{\left(\frac{A}{2} e^{j\theta}\right)}_{c_1} e^{j\Omega t} + \underbrace{\left(\frac{A}{2} e^{-j\theta}\right)}_{c_{-1}} e^{-j\Omega t}$$

Hence

$$c_n = \begin{cases} \dfrac{A}{2} e^{j\theta} & n = +1 \\ \dfrac{A}{2} e^{-j\theta} & n = -1 \\ 0 & n \neq \pm 1 \end{cases} \tag{17}$$

Perhaps because the results are so simple, students often do not realize that the phasor representation of a sinusoid is also its exponential Fourier series.

Line spectra for periodic signals

Bringing together Fourier analysis and frequency-domain representation, we point out that the exponential Fourier-series expansion of a periodic signal is a *sum of phasors*. Therefore, the line spectrum can be found directly from the series coefficients c_n. For this purpose it is advantageous to think of c_n as a complex function of the *continuous* variable f but defined only for the *discrete* values $f = nf_0$, where $f_0 = 1/T_0 = \Omega/2\pi$.

More formally, we introduce the notation $c(nf_0) = c_n$ and rewrite (3) with $\Omega = 2\pi f_0$, i.e.,

$$c(nf_0) \triangleq \frac{1}{T_0}\int_{T_0} v(t)e^{-j2\pi nf_0 t}dt \tag{18}$$

Then the exponential Fourier series becomes

$$v(t) = \sum_{n=-\infty}^{\infty} |c(nf_0)|e^{j\arg[c(nf_0)]}e^{j2\pi nf_0 t} \tag{19}$$

in which $c(nf_0)$ has been written in polar form. We interpret (19) as saying that a periodic power signal consists of a sum (usually infinite) of phasors at the frequencies $f = 0, \pm f_0, \pm 2f_0, \ldots$, the amplitude and phase of the nth component being $|c(nf_0)|$ and $\arg[c(nf_0)]$, respectively. Therefore, $|c(nf_0)|$ is the *amplitude spectrum* of $v(t)$ while $\arg[c(nf_0)]$ is the *phase spectrum*. From the previously derived properties of the Fourier series coefficients we can make several additional statements about the line spectra of periodic signals, summarized below.

1. All lines in the spectrum are located only at integer multiples of the fundamental frequency $f_0 = 1/T_0$. Hence, a periodic signal consists entirely of frequencies which are *harmonically related to* f_0.

2. The zero-frequency or DC component equals the average value of the signal, since

$$c(0) = c_0 = \frac{1}{T_0}\int_{T_0} v(t)dt$$

3. If $v(t)$ is a *real* function of time then $c(nf_0)$ has hermitian symmetry, i.e.,

$$|c(-nf_0)| = |c(nf_0)| \qquad \arg\,[c(-nf_0)] = -\arg\,[c(nf_0)]$$

which means that the amplitude and phase spectra have even and odd symmetry respectively, as observed in conjunction with Figure 6.3.

4. If the signal has *half-wave symmetry*, then

$$c(nf_0) = 0 \qquad n = 0, \pm 2, \pm 4, \ldots$$

so all the even harmonics will be absent from the line spectrum.

5. If a *real* signal has *even* symmetry in time, then $c(nf_0)$ is strictly real and hence

$$\arg\,[c(nf_0)] = 0 \qquad \text{or} \qquad \pm 180°$$

the latter being required when $c(nf_0)$ is negative. On the other hand, for a real signal with *odd* symmetry, $c(nf_0)$ is strictly imaginary and

$$\arg\,[c(nf_0)] = \pm 90°$$

since $\pm j = e^{\pm j\pi/2} = e^{\pm j90°}$.

These points, plus the mechanics of constructing line spectra, are illustrated in the following examples.

Example 6.5 Spectrum of a half-rectified sine wave

Using the results of Example 6.3, $|c(nf_0)|$ and $\arg\,[c(nf_0)]$ have been listed in Table 6.2 for the first few values of n. Negative amplitudes have been converted to phase angles of $+180°$ or $-180°$, the choice being dictated by symmetry considerations since there is no physical difference. Figure 6.11 is the resulting spectrum.

Example 6.6 Spectrum of a rectangular pulse train

To continue Example 6.2, the line spectrum of a rectangular pulse train is given by (14) when rewritten as

$$c(nf_0) = \frac{A\tau}{T_0}\,\text{sinc}\,nf_0\tau \tag{20}$$

The amplitude spectrum is then $|c(nf_0)| = (A\tau/T_0)|\text{sinc}\,nf_0\tau|$, shown in Figure 6.12*a* for the case of $\tau/T_0 = \frac{1}{5}$ so $f_0 = 1/5\tau$. This plot has been facilitated by regarding the *continuous* function $(A\tau/T_0)|\text{sinc}\,f\tau|$ as the

Table 6.2

n	c_n	$\lvert c(nf_0)\rvert$	arg $[c(nf_0)]$
0	$\frac{A}{\pi}$	$\frac{A}{\pi}$	0
± 1	$\mp j\frac{A}{4}$	$\frac{A}{4}$	$\mp 90°$
± 2	$-\frac{A}{3\pi}$	$\frac{A}{3\pi}$	$\mp 180°$
± 3	0	0	0
± 4	$-\frac{A}{15\pi}$	$\frac{A}{15\pi}$	$\mp 180°$

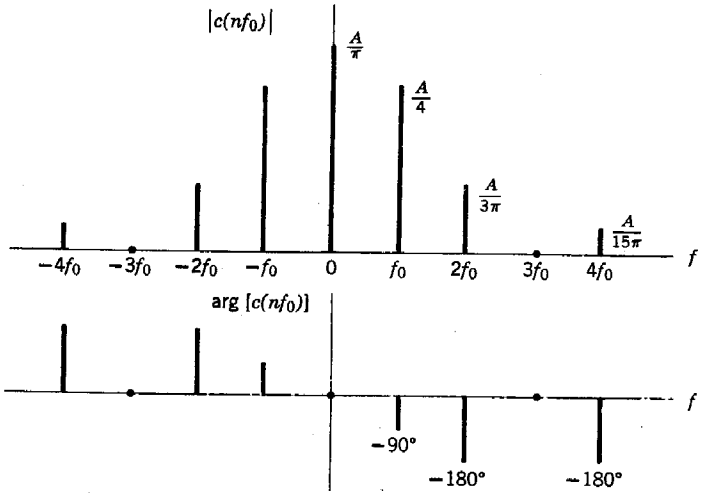

Figure 6.11. Spectrum of a half-rectified sine wave.

envelope of the amplitude lines – the dashed curve in the figure. Features to be noted here are: the uniform line spacing, save where lines are "missing" because they have zero amplitude; the even symmetry, reflecting the fact that the signal is real; and the DC component, $c(0) = A\tau/T_0$.

The phase spectrum, Figure 6.12*b*, has been constructed by noting

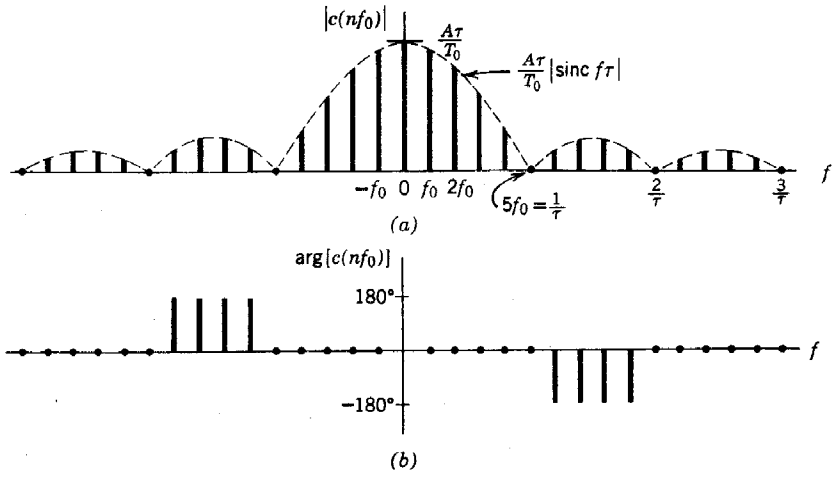

Figure 6.12. Spectrum of a rectangular pulse train, $\tau = T_0/5$.

that $c(nf_0)$ is always real but sometimes negative. Therefore, absorbing negative amplitudes in the phase, $\arg[c(nf_0)] = 0$ when $\text{sinc } nf_0\tau \geq 0$ while $\arg[c(nf_0)] = \pm 180°$ when $\text{sinc } nf_0\tau < 0$.

As a further aid to frequency-domain interpretation, the amplitude spectra and waveforms are sketched in Figure 6.13 for three values of the pulse duration τ, the pulse amplitude A and period T_0 being held fixed. When $\tau = T_0$ (Figure 6.13*a*), the signal degenerates into a constant for all time; correspondingly, $\text{sinc } nf_0\tau = \text{sinc } n = 0$ except for $n = 0$, and so the spectrum contains only one line, that line representing a DC component. This is quite logical, of course, since a constant for all time has no time variation and thus contains no frequencies other than $f = 0$.

When $\tau = T_0/2$ (Figure 6.13*b*), we have a square wave with a DC component or average value of $A/2$ and, except for the latter, the signal has half-wave symmetry. The spectrum shows this fact since the lines at $f = \pm 2f_0, \pm 4f_0, \ldots$, fall at the zero-crossing of $\text{sinc } f\tau$ and thus have zero amplitude.

Going to smaller values of pulse duration (Figure 6.13*c*), the DC component decreases – after all, the area of $v(t)$ is reduced – and the components at higher frequencies become increasingly important. Physically, these higher frequencies are required to represent the more rapid time variation of the short pulses. Thus, as the pulses are contracted in the time domain, the spectrum spreads out in the frequency domain,

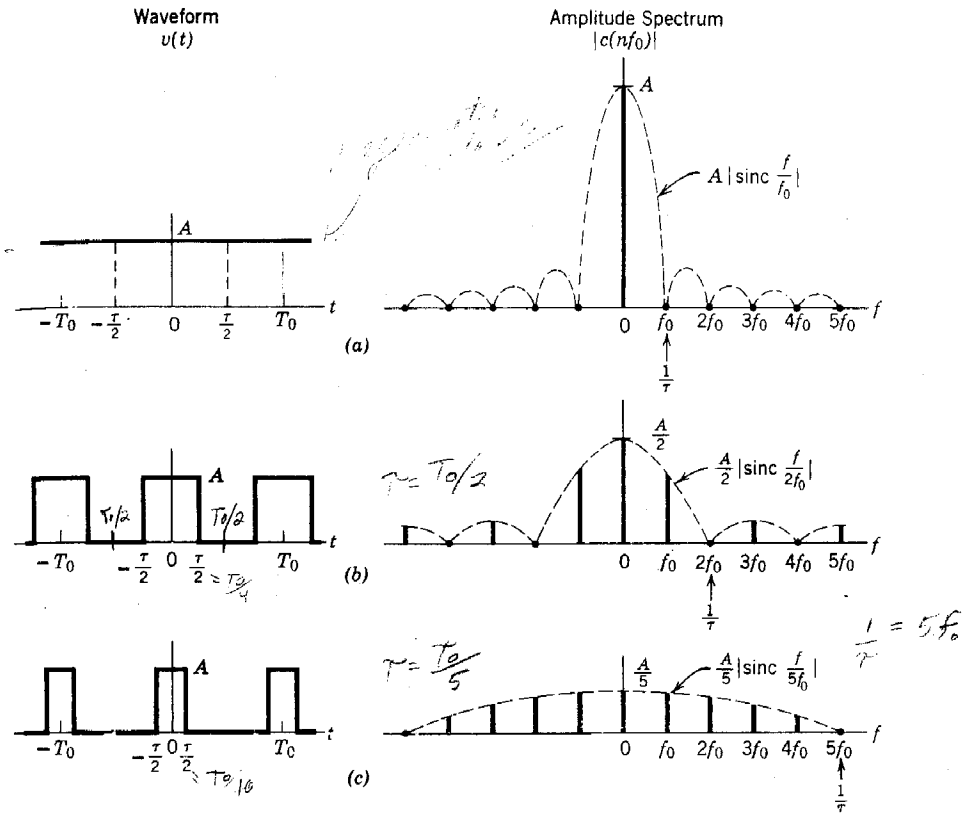

Figure 6.13. Waveform and amplitude spectrum of a rectangular pulse train. (*a*) $\tau = T_0$. (*b*) $\tau = T_0/2$. (*c*) $\tau = T_0/5$.

and vice versa. This phenomenon is known as the *reciprocal spreading* effect; it holds generally for all signals, not just for rectangular pulse trains.

6.4 FOURIER SERIES THEOREMS

There are numerous relations covering the frequency-domain effect of time-domain operations on periodic signals. Some of the more important ones are given here in the form of Fourier series theorems. These theorems are of interest for two reasons: *1.* they aid in interpreting frequency-domain properties from time-domain information, and vice versa; and

2. they are often of value as shortcuts in calculating series coefficients and line spectra.

In stating the theorems below we assume that $v(t)$, $w(t)$, and $z(t)$ are periodic signals having the *same period*, and whose spectra are $c_v(nf_0)$, $c_w(nf_0)$, and $c_z(nf_0)$, respectively.

Superposition

If α and β are constants and

$$z(t) = \alpha v(t) + \beta w(t) \tag{1a}$$

then

$$c_z(nf_0) = \alpha c_v(nf_0) + \beta c_w(nf_0) \tag{1b}$$

Hence, a linear combination in the time domain becomes a linear combination in the frequency domain. This simple and significant theorem is easily proved from the definition of $c(nf_0)$. However, because the coefficients are generally complex, care must be taken when converting (1b) to amplitude and phase spectra.

Time shift

If $z(t)$ has the same shape as $v(t)$ but delayed or shifted in time by t_d seconds so that

$$z(t) = v(t - t_d) \tag{2a}$$

where t_d may be negative as well as positive, then

$$c_z(nf_0) = c_v(nf_0)\, e^{-jn\Omega t_d} \qquad \Omega = 2\pi f_0 \tag{2b}$$

This means that *translating the time origin* affects only the *phase spectrum*, as brought out by writing (2b) in polar form

$$|c_z(nf_0)| = |c_v(nf_0)| \qquad \arg\,[c_z(nf_0)] = \arg\,[c_v(nf_0)] - n\Omega t_d$$

We prove the theorem very simply by replacing t with $t - t_d$ on both sides of Eq. (1), Sect. 6.3, giving

$$v(t - t_d) = \sum_{n=-\infty}^{\infty} c_v(nf_0)\, e^{jn\Omega(t-t_d)}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{c_v(nf_0)\, e^{-jn\Omega t_d}}_{c_z(nf_0)}\, e^{jn\Omega t}$$

Hence, if $z(t) = v(t - t_d)$ then $c_z(nf_0) = c_v(nf_0)e^{-jn\Omega t_d}$, thereby completing the proof.

Example 6.7 Spectrum of a full-rectified sine wave

Since we know the spectrum of a half-rectified sine wave, the spectrum of a full-rectified sine wave can be found directly using the superposition and time-shift theorems. Specifically, if $v(t)$ is the half-rectified wave shown in Figure 6.10, then the full-rectified wave in Figure 6.14*a* is

$$z(t) = v(t) + v\left(t - \frac{T_0}{2}\right)$$

and therefore

$$c_z(nf_0) = c_v(nf_0) + c_v(nf_0)e^{-jn\Omega T_0/2}$$

$$= c_v(nf_0)\,(1 + e^{-jn\pi}) = \begin{cases} 2c_v(nf_0) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Inserting Eq. (16), Sect. 6.3, for $c_v(nf_0)$ we finally obtain

$$c_z(nf_0) = \begin{cases} \dfrac{2A}{\pi(1-n^2)} & n = 0, \pm 2, \pm 4, \ldots \\ 0 & n = \pm 1, \pm 3, \ldots \end{cases} \tag{3}$$

and the amplitude spectrum is plotted in Figure 6.14*b*.

The fact that *all odd harmonics* are missing – including f_0 – agrees with the fact that the fundamental period of the full-rectified wave is actually $T_0/2$ instead of T_0. But perhaps more interesting is the observation that, except for the scale factor of 2, this spectrum differs from Figure 6.11 only by the absence of the first harmonic.

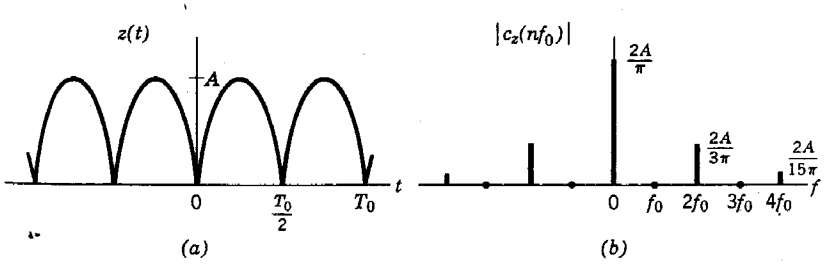

Figure 6.14. Full-rectified sine wave. (*a*) Waveform. (*b*) Amplitude spectrum.

Multiplication and modulation

Suppose that $z(t)$ is the product of two signals, $v(t)$ and $w(t)$. Its Fourier series coefficients, in terms of $c_v(nf_0)$ and $c_w(nf_0)$, are found in the following manner. Given

$$z(t) = v(t)w(t) \tag{4a}$$

we have

$$c_z(nf_0) = \frac{1}{T_0}\int_{T_0} v(t)w(t)e^{-jn\Omega t}dt$$

or, inserting the series representation for $v(t)$,

$$c_z(nf_0) = \frac{1}{T_0}\int_{T_0}\left[\sum_{m=-\infty}^{\infty} c_v(mf_0)e^{jm\Omega t}\right]w(t)e^{-jn\Omega t}dt$$

$$= \sum_{m=-\infty}^{\infty} c_v(mf_0)\underbrace{\left[\frac{1}{T_0}\int_{T_0} w(t)e^{-j(n-m)\Omega t}dt\right]}_{c_w[(n-m)f_0]}$$

where we use the different summation index m for clarity. Because the bracketed integral is just the series coefficient for $w(t)$ at frequency $(n-m)f_0$, the nth coefficient of $z(t)$ is an *infinite summation*

$$c_z(nf_0) = \sum_{m=-\infty}^{\infty} c_v(mf_0)c_w[(n-m)f_0] \tag{4b}$$

which may be recognized as a *discrete convolution*. Thus, *multiplication* in the time domain becomes *convolution* in the frequency domain.

Although (4) has conceptual value, it is not particularly useful for hand calculations unless $v(t)$ or $w(t)$ has only a few spectral lines. One such case – and an important one at that – is when $w(t) = \cos N\Omega t$ so that

$$z(t) = v(t)\cos N\Omega t \tag{5a}$$

where N is a fixed integer. From Example 6.4 it follows that

$$c_w(kf_0) = \begin{cases} \frac{1}{2} & k = \pm N \\ 0 & k \neq \pm N \end{cases}$$

and hence there are only two nonzero terms in (4b), those being for

$m = n \pm N$. Therefore, we get

$$c_z(nf_0) = \tfrac{1}{2}c_v[(n-N)f_0] + \tfrac{1}{2}c_v[(n+N)f_0] \tag{5b}$$

which says that the spectrum of $z(t)$ consists of the spectrum of $v(t)$ *translated up and down in frequency by* Nf_0 and multiplied by $\frac{1}{2}$. Because of the frequency-translation aspect and its relevance to amplitude modulation, (5) is known as the *modulation theorem.*

Average power: Parseval's theorem

The average power of a periodic signal was previously defined as

$$P = \frac{1}{T_0}\int_{T_0} |v(t)|^2 dt = \frac{1}{T_0}\int_{T_0} v(t)v^*(t)dt \tag{6}$$

Parseval's theorem relates P to the Fourier series coefficients c_n in a very simple manner, namely

$$P = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{7}$$

whose spectral interpretation is brought out by remembering that $|c_n| = |c(nf_0)|$ is the *amplitude spectrum.* Thus, the signal power is just the sum obtained by squaring and adding the heights of the amplitude lines. The fact that (7) does not involve the phase spectrum, arg $[c(nf_0)]$, reinforces our earlier remark about the dominant role of the amplitude spectrum in determining a signal's frequency content.

Further interpretation of the theorem is afforded by assuming $v(t)$ to be *real* so c_0 is real, $|c_{-n}| = |c_n|$, and (7) becomes

$$P = c_0^2 + \sum_{n=1}^{\infty} \tfrac{1}{2}|2c_n|^2 \tag{8}$$

Now, recalling the trigonometric Fourier series of Eq. (11), Sect. 6.3, we see that each sinusoidal wave of amplitude $|2c_n|$ contributes $|2c_n|^2/2$ to P. But the average power of a sinusoid having amplitude A is $A^2/2$; therefore (8) implies *superposition of average power* in that the total average power of $v(t)$ is the sum of the average powers of its sinusoidal components.

Proving Parseval's theorem is relatively routine, and will be left to the reader (Problem 6.24).