Review of LC Circuits

In a LC circuit, electrical potential energy, stored as the charge across a capacitor, and magnetic potential energy, stored as flux within an inductor, sloshes back and forth in the absence of loss, i.e., with no resistance in the circuit.

\[ E = \frac{1}{2} L I^2 \]

\[ E = \frac{1}{2} C V^2 \]

Let's consider the dynamics in detail for a sinusoidal drive voltage, \( V_{\text{drive}} = V_0 \sin (2\pi f_0 t) \).

Recall that the voltage drop across an inductor is:

\[ V(t) = L \frac{d I(t)}{dt} \]

so that the current through the inductor is just:

\[ I(t) = \frac{1}{L} \int_0^t dt' V(t') . \]

On the other hand, the current flow across a capacitor is:

\[ I(t) = C \frac{d V(t)}{dt} . \]

If we apply Kirchoff's Law for current flow to the node labeled \( V(t) \), we get:

\[ \frac{1}{L} \int_0^t dt' [V(t') - V_0 \sin(2\pi f_0 t')] + C \frac{dV(t)}{dt} = 0 . \]

Thus:

\[ \int_0^t dt' V(t') + LC \frac{dV(t)}{dt} = V_0 \frac{1}{2\pi f_0} \cos(2\pi f_0 t) . \]
or, differentiating once:

\[ V(t) + LC \frac{d^2V(t)}{dt^2} = -V_0 \sin(2\pi f_0 t). \]

Let’s stare at this. One steady-state solution is \( V(t) = A \sin(2\pi ft) \), which gives

\[ A \sin(2\pi ft) - A LC (2\pi f)^2 \sin(2\pi ft) = -V_0 \sin(2\pi f_0 t). \]

This is a valid solution so long as:

\[ f = f_0 \]

and

\[ A - A LC (2\pi f_0)^2 = -V_0, \]
or:

\[ |A| = V_0 \frac{1}{1 - LC (2\pi f_0)^2} . \]

This demonstrates resonance, i.e., the amplitude explodes when \( LC = 1/(2\pi f_0)^2 \). We identify:

\[ f_{\text{natural}} = \frac{1}{2\pi \sqrt{LC}} \]

as the natural frequency of this system. Thus the current in the circuit, \( I(t) = C \frac{dV(t)}{dt} \), is:

\[ I(t) = -\frac{V_0 2\pi f_0 C}{1 - (f_0/f_{\text{natural}})^2} \cos(2\pi f_0 t). \]

The change in sign, from “-“ when \( f_0 < f_{\text{natural}} \) to “+“ when \( f_0 > f_{\text{natural}} \), corresponds to current leading versus lagging the voltage.