

# 1 Notes on Laplace circuit analysis

## 1.1 Background

We previously learned that we can transform from the time domain to the frequency domain under steady-state conditions and thus solve algebraically for the transfer function between the input and output of a circuit. This analysis allowed us to replace inductors and capacitors by their complex impedance, as derivatives in time were replaced by  $i\omega$  and integrals in time were replaced by  $1/(i\omega)$ . The steady-state time dependence is then found by transforming back to the time domain.

The problem is that we often have a signal that turns on at a specific time, which we will take to be  $t = 0$  with no loss of generality. Can we calculate the transient behavior with a transform method, as opposed to performing a convolutional integral in the time domain? Let's recall the transform from the time domain to the frequency domain. It is given by:

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt v(t) e^{-i\omega t}. \quad (1)$$

First, what happens when  $\tilde{V}(\omega)$  cannot exist because the integral does not converge at  $t = \infty$  and/or  $t = -\infty$ ? For example, suppose  $f(t)=\text{constant}$  or worse yet a polynomial in time? One way to deal is to add an integrating factor; we chose an exponential as this will suppress any polynomial. Thus we add a factor  $\exp(-a|t|)$  to the integrand, we can take  $a \rightarrow 0$ .

Second, what happens when the system is causal, so that  $V(t)=0$  for  $t < 0$ ? Here we take the lower limit as  $t = 0$  rather than  $t = -\infty$ . This is equivalent to multiplying the integrand by a step function, denoted  $u(t)$ , where

$$u(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

All of this leads to the Laplace transform:

$$\begin{aligned} V(s) &= \int_0^{\infty} dt v(t) e^{-at} e^{-i\omega t} \\ &= \int_0^{\infty} dt v(t) e^{-st} \end{aligned} \quad (3)$$

where  $s = a + i\omega$  and  $v(t)$  is understood as  $v(t)u(t)$ . Keep in mind that the units of  $V(s)$  are Volts  $\times$  time. The inverse transform is a bit more involved, but we will show how this can be readily done for any of the functions that arise in linear circuit analysis. We have

$$v(t)u(t) = \frac{1}{2\pi i} \int_C ds V(s) e^{st}, \quad (4)$$

which is a contour integral in the complex  $s$ -plane.

All we need to know about, at least to start analyzing the kind of circuits familiar to the class, are two rules

$$\frac{dv(t)}{dt} \Rightarrow \int_0^{\infty} dt \frac{dv(t)}{dt} e^{-st} = \int_0^{\infty} dt s v(t) e^{-st} + v(t) e^{-st} \Big|_0^{\infty} = sV(s) - v(0) \quad (5)$$

$$\int_0^t dx v(x) \Rightarrow \int_0^{\infty} dt \int_0^t dx v(x) e^{-st} = \dots = \frac{1}{s} V(s) \quad (6)$$

and three transforms

$$1 \Rightarrow \int_0^{\infty} dt e^{-st} = \frac{1}{s} \quad (7)$$

$$e^{-at} \Rightarrow \int_0^{\infty} dt e^{-at} e^{-st} = \frac{1}{s+a} \quad (8)$$

$$\sin \omega t \Rightarrow \int_0^{\infty} dt \sin \omega t e^{-st} = \frac{\omega}{s^2 + \omega^2} \quad (9)$$

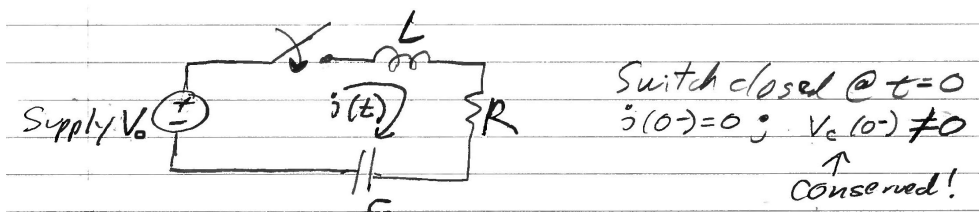
$$\cos \omega t \Rightarrow \frac{1}{\omega} \frac{d \sin(\omega t)}{dt} = \frac{s}{\omega} \frac{\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} \quad (10)$$

Note that the derivative transform includes initial conditions, as shown in the table:

Element	Ckt at $t=0^+$	Ckt at $t=\infty$

## 1.2 Application with step-to-constant input

Let's apply our new knowledge to a circuit that has a switch that closes at time  $t = 0$ . Thus the current at  $t = 0^+$  equals the current at  $t = 0^-$ , which is  $I = 0$  since the current through an inductor cannot change instantaneously. The initial voltage across the capacitor however, may not be zero. This  $V_C(0^+) = V_C(0^-)$  since the voltage across the capacitor cannot change instantaneously,



The equations are:

$$-v_o + L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t dt' i(t') + V_C(0^-) = 0. \quad (11)$$

Transforming, we get

$$\frac{-v_o}{s} + LsI(s) + RI(s) + \frac{1}{C} \frac{1}{s} I(s) + \frac{V_C(0^-)}{s} = 0. \quad (12)$$

We multiply through by  $s/L$  terms to get

$$-\frac{v_o}{L} + s^2 I(s) + \frac{R}{L} s I(s) + \frac{I(s)}{LC} + \frac{V_C(0^-)}{L} = 0 \quad (13)$$

so that

$$I(s) = \left( \frac{v_o - V_C(0^-)}{L} \right) \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \quad (14)$$

We let  $V_C(0^-) = 0$  simply to minimize the algebra in the following mathematics. Thus:

$$I(s) = \frac{v_o}{L} \frac{1}{s^2 + 2ks + \omega_o^2} \quad (15)$$

where  $k = \frac{R}{2L}$  is a decay rate and  $\omega_o = \frac{1}{\sqrt{LC}}$  is a resonant frequency. The first thing we need to do is factor the denominator. We have

$$\text{roots} = -k \pm i\sqrt{\omega_o^2 - k^2} \quad (16)$$

Thus

$$I(s) = \frac{v_o}{L} \frac{1}{(s-a)(s-a^*)} \quad (17)$$

with

$$a = -k + i\sqrt{\omega_o^2 - k^2} \quad (18)$$

and thus

$$i(t) = \frac{v_o}{L} \frac{1}{2\pi i} \int_C ds \frac{e^{st}}{(s-a)(s-a^*)}. \quad (19)$$

In order to solve this we need a refresher on the residue theorem

### "Blitz refresher on Cauchy's Residue Theorem"

Integrals in the complex plane, of the form used in linear circuit analysis, may be evaluated by

$$\int_C ds F(s) = 2\pi i \Sigma \text{Residues.}$$

When

$$\begin{aligned} F(s) &= \frac{\text{Any regular function}}{\text{Polynomial function with simple zeros}} \\ &\equiv \frac{q(s)}{p(s)} \end{aligned}$$

the residue at each zero of  $p(s)$ , or pole of  $F(s)$ , is given by the expression

$$\text{Residue} = \frac{q(s)}{\frac{\partial p(s)}{\partial s}} \Big|_{s=s_{\text{pole}}}.$$

For example, with  $q(s) = r(s)e^{st}$  and  $p(s) = (s-a)(s-b)\cdots(s-y)(s-z)$ , we have

$$\begin{aligned} \int_C ds F(s) &= \int_C ds \frac{r(s)e^{st}}{(s-a)(s-b)(s-c)\cdots(s-y)(s-z)} \\ &= 2\pi i \left[ \frac{r(s)e^{st}}{(s-b)\cdots(s-y)(s-z)} \Big|_{s=a} + \cdots + \frac{r(s)e^{st}}{(s-a)(s-b)\cdots(s-y)} \Big|_{s=z} \right] \\ &= 2\pi i \left[ \frac{r(a)e^{at}}{(a-b)\cdots(a-y)(a-z)} + \cdots + \frac{r(z)e^{zt}}{(z-a)(z-b)\cdots(z-y)} \right]. \end{aligned}$$

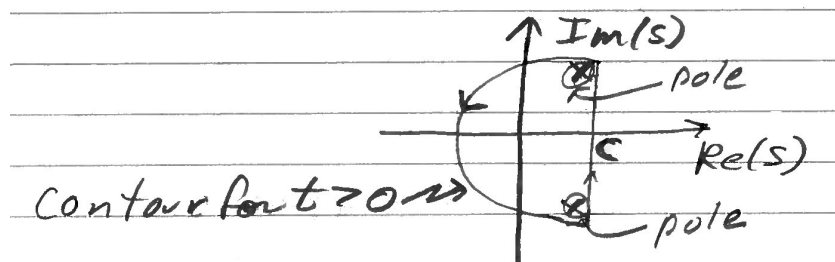
Note that complex poles always appear as conjugate pairs. Thus, for example, with

$$F(s) = \frac{e^{st}}{(s-a)(s-a^*)}$$

we find

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_C ds F(s) \\ &= \frac{1}{2\pi i} \int_C ds \frac{e^{st}}{(s-a)(s-a^*)} \\ &= \frac{1}{2\pi i} 2\pi i \left[ \frac{e^{at}}{(a-a^*)} + \frac{e^{a^*t}}{(a^*-a)} \right] \\ &= e^{\operatorname{Re}[a]t} \left[ \frac{e^{i\operatorname{Im}[a]t} - e^{-i\operatorname{Im}[a]t}}{2i\operatorname{Im}[a]} \right] \\ &= \frac{e^{\operatorname{Re}[a]t}}{\operatorname{Im}[a]} \sin(\operatorname{Im}[a]t) \end{aligned}$$

which is just the form of our solution for the prior circuit application.



The Cauchy residue theorem for the inverse transform thus yields:

$$\begin{aligned} i(t) &= \frac{v_o}{L} \frac{e^{\operatorname{Re}[a]t}}{\operatorname{Im}[a]} \sin(\operatorname{Im}[a]t) \\ &= \frac{v_o}{L} \frac{e^{-kt}}{\sqrt{\omega_o^2 - k^2}} \sin\left(\sqrt{\omega_o^2 - k^2} t\right). \end{aligned} \quad (20)$$

Note that the shift in the natural frequency, from  $\omega_o$  to  $\omega_o \left[1 - \frac{1}{2}\left(\frac{k}{\omega_o}\right)^2 + \cdots\right]$ , is quite clear. When the loss is high, i.e.,  $k > \omega_o$ , the sine term becomes a hyperbolic sine and the current just rises and decays exponentially. For the special case of  $k = \omega_o$ , so called 'critical damping',

$$i(t) = \frac{2v_o}{R} kt e^{-kt}. \quad (21)$$

Note also that the current at very short times is limited by the highest impedance, which is the induc-

tance. In particular

$$i(t) \xrightarrow{t \rightarrow 0} \frac{v_o}{L} t. \quad (22)$$

### 1.3 Application with step-to-sinusoid (tone) input

Let's now move to a more interesting dynamics and replace the source with a cosine that turns on at  $t = 0$ , that is

$$v_o(t) = v_o \cos(\omega t) \quad (23)$$

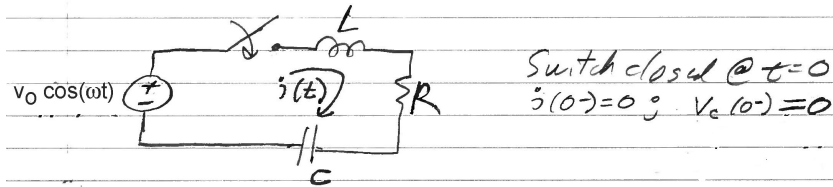
so that

$$-v_o \cos(\omega t) + L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t dt' i(t') = 0 \quad (24)$$

where, for simplicity, we take the initial voltage on the capacitor to be zero. Transforming, we get

$$\begin{aligned} I(s) &= i_o \frac{k s^2}{(s-a)(s-a^*)(s^2-\omega^2)} \\ &= i_o \frac{k s^2}{(s-a)(s-a^*)(s-i\omega)(s+i\omega)}. \end{aligned} \quad (25)$$

where we use the same abbreviations as above. We choose to use  $\cos(\omega t)u(t)$  as this reverts to  $u(t)$  as  $\omega \rightarrow 0$ .



The circuit will respond at both the driven frequency and at the natural frequency. We have, from the residue theorem, four terms that we will evaluate in pairs, i.e.

$$i(t) = i_o [f_1(t) + f_2(t)] \quad (26)$$

where

$$f_1(t) = \frac{k s^2 e^{st}}{(s-a^*)(s^2+\omega^2)} \Big|_{s=a} + \frac{k s^2 e^{st}}{(s-a)(s^2+\omega^2)} \Big|_{s=a^*} \quad (27)$$

and

$$f_2(t) = \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s+i\omega)} \Big|_{s=i\omega} + \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s-i\omega)} \Big|_{s=-i\omega}. \quad (28)$$

Before we solder on with  $f_1(t)$ , let's calculate some of the algebraic terms that we will need, i.e.,

$$a - a^* = i2\sqrt{\omega_o^2 - k^2}, \quad (29)$$

$$aa^* = \omega_o^2, \quad (30)$$

$$a^2 = 2k^2 - \omega_o^2 + i2\sqrt{\omega_o^2 - k^2}. \quad (31)$$

and

$$a^{*2} = 2k^2 - \omega_o^2 - i2\sqrt{\omega_o^2 - k^2}. \quad (32)$$

$$f_1(t) = \frac{ke^{-kt}}{\sqrt{\omega_o^2 - k^2}} \frac{1}{2i} \left( \frac{2k^2 - \omega_o^2 + i2k\sqrt{\omega_o^2 - k^2}}{2k^2 + \omega^2 - \omega_o^2 + i2k\sqrt{\omega_o^2 - k^2}} e^{i\sqrt{\omega_o^2 - k^2} t} - \frac{2k^2 - \omega_o^2 - i2k\sqrt{\omega_o^2 - k^2}}{2k^2 + \omega^2 - \omega_o^2 - i2k\sqrt{\omega_o^2 - k^2}} e^{-i\sqrt{\omega_o^2 - k^2} t} \right). \quad (33)$$

We rationalize the denominator, noting that

$$\left(2k^2 + \omega^2 - \omega_o^2 + i2k\sqrt{\omega_o^2 - k^2}\right) \left(2k^2 + \omega^2 - \omega_o^2 - i2k\sqrt{\omega_o^2 - k^2}\right) = (\omega^2 - \omega_o^2)^2 + (2k\omega)^2, \quad (34)$$

$$\left(2k^2 - \omega_o^2 + i2k\sqrt{\omega_o^2 - k^2}\right) \left(2k^2 + \omega^2 - \omega_o^2 - i2k\sqrt{\omega_o^2 - k^2}\right) = \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) + i2k\omega^2\sqrt{\omega_o^2 - k^2} \quad (35)$$

and

$$\left(2k^2 - \omega_o^2 - i2k\sqrt{\omega_o^2 - k^2}\right) \left(2k^2 + \omega^2 - \omega_o^2 + i2k\sqrt{\omega_o^2 - k^2}\right) = \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) - i2k\omega^2\sqrt{\omega_o^2 - k^2} \quad (36)$$

so that

$$\begin{aligned} f_1(t) &= \frac{ke^{-kt}}{\sqrt{\omega_o^2 - k^2}[(\omega^2 - \omega_o^2)^2 + (2k\omega)^2]} \quad (37) \\ &\times \left( \left[ \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) \right] \frac{e^{i\sqrt{\omega_o^2 - k^2} t} - e^{-i\sqrt{\omega_o^2 - k^2} t}}{2i} + 2k\omega^2\sqrt{\omega_o^2 - k^2} \frac{e^{i\sqrt{\omega_o^2 - k^2} t} + e^{-i\sqrt{\omega_o^2 - k^2} t}}{2} \right) \\ &= \frac{ke^{-kt}}{\sqrt{\omega_o^2 - k^2}[(\omega^2 - \omega_o^2)^2 + (2k\omega)^2]} \\ &\times \left( \left[ \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) \right] \sin(\sqrt{\omega_o^2 - k^2} t) + 2k\omega^2\sqrt{\omega_o^2 - k^2} \cos(\sqrt{\omega_o^2 - k^2} t) \right) \\ &= \frac{\omega_o^2 ke^{-kt}}{\sqrt{(\omega_o^2 - k^2)[(\omega^2 - \omega_o^2)^2 + (2k\omega)^2]}} \\ &\times \left( \frac{\omega_o^4 - \omega^2(\omega_o^2 - 2k^2)}{\omega_o^2\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \sin(\sqrt{\omega_o^2 - k^2} t) + \frac{2k\omega^2\sqrt{\omega_o^2 - k^2}}{\omega_o^2\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \cos(\sqrt{\omega_o^2 - k^2} t) \right). \end{aligned}$$

The weighting factors for the sine and cosine terms satisfy the right triangle rule

$$\left[ \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) \right]^2 + \left[ 2k\omega^2\sqrt{\omega_o^2 - k^2} \right]^2 = \left[ \omega_o^2\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2} \right]^2 \quad (38)$$

so that with the definition

$$\phi_1(\omega) = \text{atan} \left( \frac{2k\omega^2\sqrt{\omega_o^2 - k^2}}{\omega_o^4 - \omega^2(\omega_o^2 - 2k^2)} \right) \quad (39)$$

we have

$$\begin{aligned} f_1(t) &= \frac{\omega_o^2 ke^{-kt}}{\sqrt{(\omega_o^2 - k^2)[(\omega^2 - \omega_o^2)^2 + (2k\omega)^2]} \quad (40) \\ &\times \left( \cos[\phi_1(\omega)] \sin(\sqrt{\omega_o^2 - k^2} t) + \sin[\phi_1(\omega)] \cos(\sqrt{\omega_o^2 - k^2} t) \right) \\ &= \frac{\omega_o^2}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \frac{ke^{-kt}}{\sqrt{(\omega_o^2 - k^2)}} \sin \left( \sqrt{\omega_o^2 - k^2} t + \phi_1(\omega) \right). \end{aligned}$$

The first term is maximized for the choice of drive frequency  $\omega = \sqrt{\omega_o^2 - 2k^2}$ , which is slightly lower than the natural response frequency of  $\omega = \sqrt{\omega_o^2 - k^2}$ . Lastly, and as a sanity check, in the limit of  $\omega \rightarrow 0$ , we recover the result for the response to a step input, i.e.

$$f_1(t) \xrightarrow{\omega \rightarrow 0} \frac{ke^{-kt}}{\sqrt{\omega_o^2 - k^2}} \sin \left( \sqrt{\omega_o^2 - k^2} t \right). \quad (41)$$

Let's now move on to the driven term  $f_2(t)$ . We first note the evaluations:

$$(s - a)(s - a^*)|_{s=i\omega} = (s^2 + 2ks + \omega_o^2)|_{s=i\omega} = (\omega_o^2 - \omega^2) + i2k\omega, \quad (42)$$

$$(s - a)(s - a^*)|_{s=-i\omega} = (s^2 + 2ks + \omega_o^2)|_{s=-i\omega} = (\omega_o^2 - \omega^2) - i2k\omega. \quad (43)$$

and

$$[(\omega_o^2 - \omega^2) + i2k\omega][(\omega_o^2 - \omega^2) - i2k\omega] = (\omega_o^2 - \omega^2)^2 + (2k\omega)^2 \quad (44)$$

Then

$$\begin{aligned} f_2(t) &= \frac{k(i\omega)^2 e^{i\omega t}}{[(\omega_o^2 - \omega^2) + i2k\omega](i2\omega)} + \frac{k(-i\omega)^2 e^{-i\omega t}}{[(\omega_o^2 - \omega^2) - i2k\omega](-i2\omega)} \quad (45) \\ &= \frac{-k\omega}{2i} \left( \frac{e^{i\omega t}[(\omega_o^2 - \omega^2) - i2k\omega] - e^{-i\omega t}[(\omega_o^2 - \omega^2) + i2k\omega]}{(\omega_o^2 - \omega^2)^2 + (2k\omega)^2} \right) \\ &= \frac{k\omega}{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2} \left( 2k\omega \frac{e^{i\omega t} + e^{-i\omega t}}{2} + (\omega^2 - \omega_o^2) \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) \\ &= \frac{k\omega}{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2} ((\omega^2 - \omega_o^2) \sin(\omega t) + 2k\omega \cos(\omega t)) \\ &= \frac{k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \left( \frac{\omega^2 - \omega_o^2}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \sin(\omega t) + \frac{2k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \cos(\omega t) \right) \end{aligned}$$

The weighting factors for the sine and cosine terms satisfy the right triangle rule, so

$$\begin{aligned} f_2(t) &= \frac{k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} (\sin[\phi_2(\omega)]\sin(\omega t) + \cos[\phi_2(\omega)]\cos(\omega t)) \quad (46) \\ &= \frac{k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \cos(\omega t - \phi_2(\omega)) \end{aligned}$$

where

$$\phi_2(\omega) = \text{atan} \left( \frac{\omega^2 - \omega_o^2}{2k\omega} \right) \quad (47)$$

$$\begin{aligned} i(t) &= \frac{v_o}{2kL} (f_1(t) + f_2(t)) \quad (48) \\ &= \frac{v_o}{2\omega_o L} \frac{\omega_o^2}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \left( \frac{\omega_o e^{-kt}}{\sqrt{(\omega_o^2 - k^2)}} \sin(\sqrt{\omega_o^2 - k^2} t + \phi_1(\omega)) + \frac{\omega}{\omega_o} \cos(\omega t - \phi_2(\omega)) \right). \end{aligned}$$

At  $t = 0^+$ , the amplitude of the response is

$$i(0^+) = \frac{v_o}{R} \frac{(2k\omega)^2}{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2} \quad (49)$$