1 Notes on Laplace circuit analysis

1.1 Background

We previously learned that we can transform from the time domain to the frequency domain under steady-state conditions and thus solve algebraically for the transfer function between the input and output of a circuit. This analysis allowed us to replace inductors and capacitors by their complex impedance, as derivatives in time were replaced by $i\omega$ and integrals in time were replaced by $1/(i\omega)$. The steady-state time dependence is then found by transforming back to the time domain.

The problem is that we often have a signal that turns on at a specific time, which we will take to be $t = 0$ with no loss of generality. Can we calculate the transient behavior with a transform method, as opposed to performing a convolutional integral in the time domain? Let’s recall the transform from the time domain to the frequency domain. It is given by:

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt \ v(t) \ e^{-i\omega t}. \quad (1)$$

First, what happens when $\tilde{V}(\omega)$ cannot exist because the integral does not converge at $t = \infty$ and/or $t = -\infty$? For example, suppose $f(t) = \text{constant}$ or worse yet a polynomial in time? One way to deal is to add an integrating factor; we chose an exponential as this will suppress any polynomial. Thus we add a factor $\exp(-a|t|)$ to the integrand, we can take $a \to 0$.

Second, what happens when the system is causal, so that $V(t) = 0$ for $t < 0$? Here we take the lower limit as $t = 0$ rather than $t = -\infty$. This is equivalent to multiplying the integrand by a step function, denoted $u(t)$, where

$$u(t) = \begin{cases} 
1, & t > 0, \\
0, & \text{otherwise},
\end{cases} \quad (2)$$

All of this leads to the Laplace transform:

$$V(s) = \int_{0}^{\infty} dt \ v(t) \ e^{-st} e^{-i\omega t} \quad (3)$$

where $s = a + i\omega$ and $v(t)$ is understood as $v(t)u(t)$. Keep in mind that the units of $V(s)$ are Volts $\times$ time. The inverse transform is a bit more involved, but we will show how this can be readily done for any of the functions that arise in linear circuit analysis. We have

$$v(t)u(t) = \frac{1}{2\pi i} \int_{C} ds \ V(s) \ e^{st}, \quad (4)$$
which is a contour integral in the complex $s$-plane.

All we need to know about, at least to start analyzing the kind of circuits familiar to the class, are two rules

$$\frac{dv(t)}{dt} \Rightarrow \int_0^\infty dt \frac{dv(t)}{dt} e^{-st} = \int_0^\infty dt \ s \ v(t) \ e^{-st} + v(t) \ e^{-st}|_0^\infty = sV(s) - v(0) \quad (5)$$

$$\int_0^t dx \ v(x) \Rightarrow \int_0^\infty dt \ \int_0^t dx \ v(x) \ e^{-st} = \cdots = \frac{1}{s} V(s) \quad (6)$$

and three transforms

$$1 \Rightarrow \int_0^\infty dt \ e^{-st} = \frac{1}{s} \quad (7)$$

$$e^{-at} \Rightarrow \int_0^\infty dt \ e^{-at} e^{-st} = \frac{1}{s + a} \quad (8)$$

$$\sin\omega t \Rightarrow \int_0^\infty dt \ \sin\omega t \ e^{-st} = \frac{\omega}{s^2 + \omega^2} \quad (9)$$

$$\cos\omega t \Rightarrow \frac{1}{\omega} \frac{d \sin(\omega t)}{dt} = \frac{s \ \omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} \quad (10)$$

Note that the derivative transform includes initial conditions, as shown in the table:

<table>
<thead>
<tr>
<th>Element</th>
<th>Ch + s t $=0$</th>
<th>Ch + s t $=\infty$</th>
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<tbody>
<tr>
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### 1.2 Application with step-to-constant input

Let’s apply our new knowledge to a circuit that has a switch that closes at time $t = 0$. Thus the current at $t = 0^+$ equals the current at $t = 0^-$, which is $I = 0$ since the current through an inductor cannot change instantaneously. The initial voltage across the capacitor however, may not be zero. This $V_C(0^+) = V_C(0^-)$ since the voltage across the capacitor cannot change instantaneously,
The equations are:

\[-v_0 + L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t dt' i(t') + V_C(0^-) = 0. \quad (11)\]

Transforming, we get

\[-\frac{v_0}{s} + LsI(s) + RI(s) + \frac{1}{s} I(s) + \frac{V_C(0^-)}{s} = 0. \quad (12)\]

We multiply through by $\frac{s}{L}$ terms to get

\[-\frac{v_0}{L} + s^2 I(s) + \frac{R}{L} s I(s) + \frac{I(s)}{LC} + \frac{V_C(0^-)}{L} = 0 \quad (13)\]

so that

\[I(s) = \left(\frac{v_0 - V_C(0^-)}{L}\right) \frac{1}{s^2 + \frac{R}{L} s + \frac{1}{LC}}. \quad (14)\]

We let $V_C(0^-) = 0$ simply to minimize the algebra in the following mathematics. Thus:

\[I(s) = \frac{v_0}{L} \frac{1}{s^2 + \frac{2ks + \omega_o^2}{L}} \quad (15)\]

where $k = \frac{R}{2L}$ is a decay rate and $\omega_o = \frac{1}{\sqrt{LC}}$ is a resonant frequency. The first thing we need to do is factor the denominator. We have

roots $= -k \pm i \sqrt{\omega_o^2 - k^2} \quad (16)$

Thus

\[I(s) = \frac{v_0}{L} \frac{1}{(s-a)(s-a^*)} \quad (17)\]

with

\[a = -k + i \sqrt{\omega_o^2 - k^2} \quad (18)\]

and thus

\[i(t) = \frac{v_0}{L} \frac{1}{2\pi i} \int_C ds \frac{e^{st}}{(s-a)(s-a^*)}. \quad (19)\]

In order to solve this we need a refresher on the residue theorem

"Blitz refresher on Cauchy’s Residue Theorem"

Integrals in the complex plane, of the form used in linear circuit analysis, may be evaluated by

\[\int_C ds F(s) = 2\pi i \Sigma \text{Residues.} \]

When

\[F(s) = \text{Any regular function} \]

\[= \frac{q(s)}{p(s)} \]

the residue at each zero of $p(s)$, or pole of $F(s)$, is given by the expression

\[\text{Residue} = \frac{q(s)}{\frac{\partial q(s)}{\partial s}} \bigg|_{s = \text{pole}}.\]
For example, with \( q(s) = r(s)e^{st} \) and \( p(s) = (s - a)(s - b) \cdots (s - y)(s - z) \), we have

\[
\int_C ds F(s) = \int_C \frac{r(s)e^{st}}{(s - a)(s - b)(s - c) \cdots (s - y)(s - z)} ds
\]

\[
= 2\pi i \left[ \frac{r(s)e^{st}}{(s - b) \cdots (s - y)(s - z)} \bigg|_{s = a} + \cdots + \frac{r(s)e^{st}}{(s - a)(s - b) \cdots (s - y)} \bigg|_{s = z} \right]
\]

\[
= 2\pi i \left[ \frac{r(a)e^{at}}{(a - b) \cdots (a - y)(a - z)} + \cdots + \frac{r(z)e^{zt}}{(z - a)(z - b) \cdots (z - y)} \right].
\]

Note that complex poles always appear as conjugate pairs. Thus, for example, with

\[
F(s) = \frac{e^{st}}{(s - a)(s - a^*)}
\]

we find

\[
f(t) = \frac{1}{2\pi i} \int_C ds F(s)
\]

\[
= \frac{1}{2\pi i} \int_C \frac{e^{st}}{(s - a)(s - a^*)} ds
\]

\[
= \frac{1}{2\pi i} 2\pi i \left[ \frac{e^{at}}{(a - a^*)} + \frac{e^{a^*t}}{(a^* - a)} \right]
\]

\[
= e^{Re[a]t} \left[ \frac{e^{i\text{Im}[a]t} - e^{-i\text{Im}[a]t}}{2i\text{Im}[a]} \right]
\]

\[
= \frac{e^{Re[a]t}}{\text{Im}[a]} \sin(\text{Im}[a]t)
\]

which is just the form of our solution for the prior circuit application.

\[
\begin{array}{c}
\text{Contour} \quad \omega \quad \text{pole} \\
\text{Re}(s) \quad \text{Im}(s)
\end{array}
\]

The Cauchy residue theorem for the inverse transform thus yields:

\[
i(t) = \frac{v_o}{L} \left( \frac{e^{Re[a]t}}{\text{Im}[a]} \sin(\text{Im}[a]t) \right)
\]

\[
= \frac{v_o}{L} \frac{e^{-kt}}{\sqrt{\omega_o^2 - k^2}} \sin \left( \sqrt{\omega_o^2 - k^2} t \right).
\]

Note that the shift in the natural frequency, from \( \omega_o \) to \( \omega_o \left[ 1 - \frac{1}{2} \left( \frac{k}{\omega_o} \right)^2 + \cdots \right] \), is quite clear. When the loss is high, i.e., \( k > \omega_o \), the sine term becomes a hyperbolic sine and the current just rises and decays exponentially. For the special case of \( k = \omega_o \), so called ‘critical damping’,

\[
i(t) = \frac{2v_o}{R} kt e^{-kt}.
\]

Note also that the current at very short times is limited by the highest impedance, which is the induc-
tance. In particular
\[ i(t) \xrightarrow{\ t\to0\ } \frac{v_o}{L} t. \]  

1.3 Application with step-to-sinusoid (tone) input

Let’s now move to a more interesting dynamics and replace the source with a cosine that turns on at \( t = 0 \), that is
\[ v_o(t) = v_o \cos(\omega t) \]  
so that
\[ -v_o \cos(\omega t) + L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t dt' i(t') = 0 \]  
where, for simplicity, we take the initial voltage on the capacitor to be zero. Transforming, we get
\[ I(s) = i_o \frac{k s^2}{(s-a)(s-a^*)(s^2 - \omega^2)} \]  
where we use the same abbreviations as above. We choose to use \( \cos(\omega t)u(t) \) as this reverts to \( u(t) \) as \( \omega \to 0 \).

The circuit will respond at both the driven frequency and at the natural frequency. We have, from the residue theorem, four terms that we will evaluate in pairs, i.e.
\[ i(t) = i_o [f_1(t) + f_2(t)] \]  
where
\[ f_1(t) = \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s^2 + \omega^2)} \bigg|_{s=a} + \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s^2 + \omega^2)} \bigg|_{s=a^*} \]  
and
\[ f_2(t) = \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s+i\omega)} \bigg|_{s=\omega} + \frac{k s^2 e^{st}}{(s-a)(s-a^*)(s-i\omega)} \bigg|_{s=-i\omega}. \]  

Before we solder on with \( f_1(t) \), let’s calculate some of the algebraic terms that we will need, i.e.,
\[ a - a^* = i2\sqrt{\omega_o^2 - k^2}, \]  
\[ aa^* = \omega_o^2, \]  
\[ a^2 = 2k^2 - \omega_o^2 + i2\sqrt{\omega_o^2 - k^2}. \]  
and
\[ a^*^2 = 2k^2 - \omega_o^2 - i2\sqrt{\omega_o^2 - k^2}. \]
\[ f_1(t) = \frac{ke^{-kt}}{\sqrt{w_o^2 - k^2}} \left( \frac{2k^2 - \omega^2 + i2k \sqrt{w_o^2 - k^2}}{2k^2 + \omega^2 - w_o^2 + i2k \sqrt{w_o^2 - k^2}} e^{i\sqrt{w_o^2 - k^2} t} - \frac{2k^2 - \omega^2 - i2k \sqrt{w_o^2 - k^2}}{2k^2 + \omega^2 - w_o^2 - i2k \sqrt{w_o^2 - k^2}} e^{-i\sqrt{w_o^2 - k^2} t} \right). \]  

(33)

We rationalize the denominator, noting that

\[ \left( 2k^2 + \omega^2 - w_o^2 + i2k \sqrt{w_o^2 - k^2} \right) \left( 2k^2 + \omega^2 - w_o^2 - i2k \sqrt{w_o^2 - k^2} \right) = (\omega^2 - w_o^2)^2 + (2k\omega)^2, \]
\[ (34) \]

\[ \left( 2k^2 - w_o^2 + i2k \sqrt{w_o^2 - k^2} \right) \left( 2k^2 + \omega^2 - w_o^2 - i2k \sqrt{w_o^2 - k^2} \right) = \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) + i2k\omega^2 \sqrt{w_o^2 - k^2} \]
\[ (35) \]

and

\[ \left( 2k^2 - w_o^2 - i2k \sqrt{w_o^2 - k^2} \right) \left( 2k^2 + \omega^2 - w_o^2 + i2k \sqrt{w_o^2 - k^2} \right) = \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) - i2k\omega^2 \sqrt{w_o^2 - k^2} \]
\[ (36) \]

so that

\[ f_1(t) = \frac{ke^{-kt}}{\sqrt{w_o^2 - k^2}[\omega^2 - w_o^2]^2 + (2k\omega)^2] \times \left[ \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) \right] e^{i\sqrt{w_o^2 - k^2} t} - \frac{e^{-i\sqrt{w_o^2 - k^2} t}}{2} + 2k\omega^2 \sqrt{w_o^2 - k^2} e^{i\sqrt{w_o^2 - k^2} t} + e^{-i\sqrt{w_o^2 - k^2} t} \right] \]
\[ (37) \]

The weighting factors for the sine and cosine terms satisfy the right triangle rule

\[ \left[ \omega_o^4 - \omega^2(\omega_o^2 - 2k^2) \right]^2 + \left[ 2k\omega^2 \sqrt{w_o^2 - k^2} \right]^2 = \left[ \omega_o^2 \sqrt{(\omega^2 - w_o^2)^2 + (2k\omega)^2} \right]^2 \]
\[ (38) \]

so that with the definition

\[ \phi_1(\omega) = \arctan \left( \frac{2k\omega^2 \sqrt{w_o^2 - k^2}}{\omega_o^4 - \omega^2(\omega_o^2 - 2k^2)} \right) \]
\[ (39) \]

we have

\[ f_1(t) = \frac{\omega_o^2 ke^{-kt}}{\sqrt{(\omega_o^2 - k^2)[(\omega^2 - w_o^2)^2 + (2k\omega)^2]}} \times \left( \cos[\phi_1(\omega)] \sin(\sqrt{w_o^2 - k^2} t) + \sin[\phi_1(\omega)] \cos(\sqrt{w_o^2 - k^2} t) \right) \]
\[ = \frac{\omega_o^2}{\sqrt{(\omega^2 - w_o^2)^2 + (2k\omega)^2}} \frac{ke^{-kt}}{\sqrt{(\omega_o^2 - k^2)^2 + (2k\omega)^2}} \sin \left( \sqrt{w_o^2 - k^2} t + \phi_1(\omega) \right). \]
\[ (40) \]

The first term is maximized for the choice of drive frequency \( \omega = \sqrt{\omega_o^2 - 2k^2} \), which is slightly lower than the natural response frequency of \( \omega = \sqrt{\omega_o^2 - k^2} \). Lastly, and as a sanity check, in the limit of \( \omega \to 0 \), we recover the result for the response to a step input, i.e.

\[ f_1(t) \overset{\omega \to 0}{=} \frac{ke^{-kt}}{\sqrt{\omega_o^2 - k^2}} \sin \left( \sqrt{w_o^2 - k^2} t \right). \]
\[ (41) \]
Let’s now move on to the driven term \( f_2(t) \). We first note the evaluations:

\[
(s - a)(s - a^*)|_{s = i\omega} = (s^2 + 2k\omega + \omega_o^2)|_{s = i\omega} = (\omega_o^2 - \omega^2) + i2k\omega, \tag{42}
\]

\[
(s - a)(s - a^*)|_{s = -i\omega} = (s^2 + 2k\omega + \omega_o^2)|_{s = -i\omega} = (\omega_o^2 - \omega^2) - i2k\omega. \tag{43}
\]

and

\[
[(\omega_o^2 - \omega^2) + i2k\omega][(\omega_o^2 - \omega^2) - i2k\omega] = (\omega_o^2 - \omega^2)^2 + (2k\omega)^2 \tag{44}
\]

Then

\[
f_2(t) = \frac{k (i\omega)^2 e^{i\omega t}}{[(\omega_o^2 - \omega^2) + i2k\omega][i2\omega]} + \frac{k (-i\omega)^2 e^{-i\omega t}}{[(\omega_o^2 - \omega^2) - i2k\omega][-i2\omega]}
\]

\[
= k\omega \left( \frac{e^{i\omega t}[(\omega_o^2 - \omega^2) - i2k\omega] - e^{-i\omega t}[(\omega_o^2 - \omega^2) + i2k\omega]}{(\omega_o^2 - \omega^2)^2 + (2k\omega)^2} \right)
\]

\[
= \frac{2k\omega}{(\omega_o^2 - \omega^2)^2 + (2k\omega)^2} \left( \frac{\omega^2 - \omega_o^2}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \sin(\omega t) + \frac{2k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \cos(\omega t) \right)
\]

The weighting factors for the sine and cosine terms satisfy the right triangle rule, so

\[
f_2(t) = \frac{k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \sin(\phi_2(\omega)) \sin(\omega t) + \cos(\phi_2(\omega)) \cos(\omega t) \tag{46}
\]

\[
= \frac{k\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \cos(\omega t - \phi_2(\omega))
\]

where

\[
\phi_2(\omega) = \arctan \left( \frac{\omega^2 - \omega_o^2}{2k\omega} \right) \tag{47}
\]

\[
i(t) = \frac{v_o}{2kL} (f_1(t) + f_2(t)) \tag{48}
\]

\[
= \frac{v_o}{2\omega_o L} \frac{\omega_o^2}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2}} \left( \frac{\omega_o e^{-kt}}{\sqrt{\omega_o^2 - k^2}} \sin \left( \sqrt{\omega_o^2 - k^2} t + \phi_1(\omega) \right) + \frac{\omega}{\omega_o} \cos(\omega t - \phi_2(\omega)) \right).
\]

At \( t = 0^+ \), the amplitude of the response is

\[
i(0^+) = \frac{v_o}{R} \frac{(2k\omega)^2}{(\omega^2 - \omega_o^2)^2 + (2k\omega)^2} \tag{49}
\]