Notes on Boolean Operations - Physics 120 - David Kleinfeld - 2017

Addition of two bits serves to illustrate XOR and AND:

0	1	0	1
0	0	1	1
00	01	01	10

The 1's place is the logic function Exclusive OR (XOR) while the 2's place is the logic function AND, i.e.,

X	Y	AND
0	0	0
0	1	0
1	0	0
1	1	1

and

X	Y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

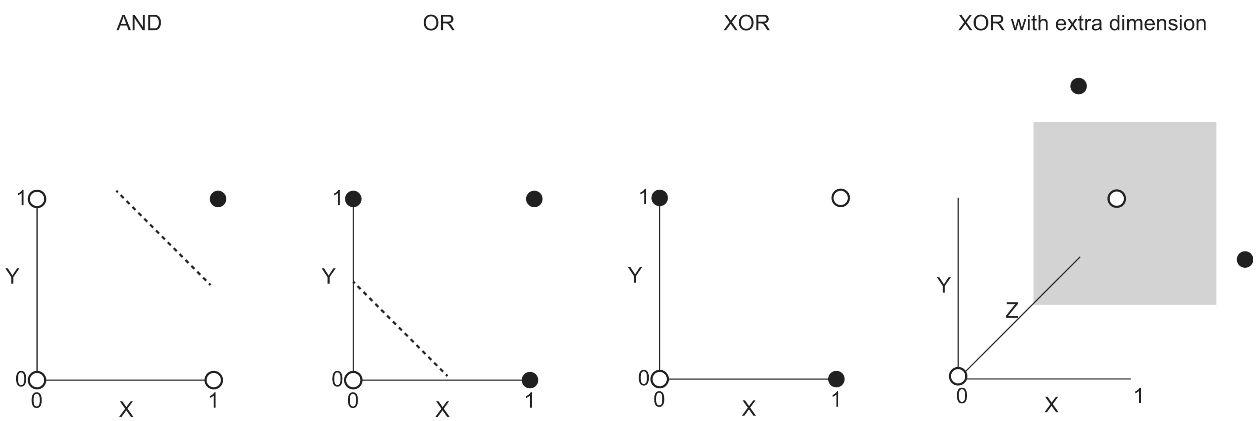
, along with

X	Y	OR
0	0	0
0	1	1
1	0	1
1	1	1

and

X	$\bar{X}$
0	1
1	0

where the bar means "not". Plots of AND sand OR show that they can be computed by dividing the part of the plane with 0's from that with 1's with a single line. This is not true for XOR; we need to add a dimension so that a plane can be used to separate the 0's from the 1's. This will requires two levels of logic.



For simplicity of notation, we write X AND Y as XY and X OR Y as X+Y.

Identities:
$0+X = X$
$1+X = 1$
$0X = 0$
$1X = 1$

Association:
$X+(Y+Z) = (X+Y)+Z = (X+Z)+Y$

Distribution:
$X(Y+Z) = XY+XZ$
$X+(YZ) = (X+Y)(X+Z)$

Absorption:
$X+(XY) = X$
$X+(\bar{X}Y) = X+Y$
$X(X+Y) = X$
$X(\bar{X}+Y) = XY$

DeMorgan:
$\overline{X+Y} = \bar{X}\,\bar{Y}$
$\overline{XY} = \bar{X}+\bar{Y}$
$\bar{\bar{X}} = X$

Any Boolean function can be written as the OR of multiple ANDs, that is, can be computed with two levels of logic albeit with arbitrarily large fan outs from the first layer of ANDs and fan ins to the second layer of ORs. Thus:

X_1	X_2	X_3	X_4	$\cdots$	X_N	F
1	1	0	1	$\cdots$	0	1
0	1	0	1	$\cdots$	0	0
0	0	0	0	$\cdots$	1	0
0	1	0	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$

so that $F = (X_1X_2\bar{X}_3X_4 \cdots \bar{X}_N) + (\bar{X}_1X_2\bar{X}_3X_4 \cdots X_N) + \cdots$. Thus, looking back, $XOR = \bar{Y}X + Y\bar{X}$.