Notes on Relaxation Oscillators
Physics 120, David Kleinfeld, Spring 2017

This is a basic oscillator circuit in which the voltage across a capacitor relaxes toward a time varying target voltage. Consider the circuit below, using a high gain op-amp as a comparator:

The time-varying voltage at \( V_+ \) serves as a target level that is a fraction \( R_2/(R_1+R_2) \) of the output voltage \( V_{out} \). The time-varying voltage at \( V_- \) heads toward this value and when it exceeds the value, the output of the amplifier changes sign, thus the target changes sign, and the voltage at \( V_- \) heads toward this new value.

We know that \( V_{out} = A(V_+ - V_-) \) where \( A \gg 1 \).

Since \( V_{out} \) is bounded by \( V_{supply} \), this means that

\[
V_{out} = \begin{cases} 
  +V_{supply} & V_+ > V_- \\
  -V_{supply} & V_- < V_+ 
\end{cases}
\]

The voltage divider assures that the value of \( V_- \), which can asymptote at \( V_- = V_{out} \), will be compared to a value for \( V_+ \) that is smaller than \( V_{out} \), i.e., \( V_- = \frac{R_2}{R_1+R_2} V_{out} \).

\( V_- (t) \) will evolve in time according to:

\[
C \frac{dV_-}{dt} + \frac{V_- - V_{out}}{R} = 0
\]

with \( \tau \equiv RC \). Thus:

\[
\frac{dV_-}{dt} + \frac{1}{\tau} V_- = \frac{1}{\tau} V_{out}
\]
The solution to this homogeneous part of this equation is:

\[ V_-(t) = V_0 e^{-t/\tau}. \]

so that the full solution is:

\[ V_-(t) = V_0 e^{-t/\tau} + \int_0^t \left( \frac{1}{\tau} V_{\text{out}} \right) e^{-(t-x)/\tau} \, dx. \]

We take \( t = 0 \) as the time of the last transition and consider the interval of time up to the next transition, i.e., \( t = T/2 \), so that \( V_{\text{out}} \) is a constant. The value of \( V_0 \) just after the transition, i.e., \( V_0(0^+) \), is opposite in sign to that of \( V_{\text{out}} \) so

\[ V_0(0) = -\frac{R_2}{R_1 + R_2} V_{\text{out}}. \]

Thus:

\[ V_-(t) = -\frac{R_2}{R_1 + R_2} V_{\text{out}} e^{-t/\tau} + V_{\text{out}} (1 - e^{-t/\tau}) \]

At \( t = T/2 \) the value of \( V_- \) will reach the threshold level

\[ V_-(T/2) = +\frac{R_2}{R_1 + R_2} V_{\text{out}} \]

so

\[ \frac{R_2}{R_1 + R_2} V_{\text{out}} = -\frac{R_2}{R_1 + R_2} V_{\text{out}} e^{-T/2\tau} + V_{\text{out}} \left(1 - e^{-T/2\tau}\right) \]

or

\[ T = 2 \tau \log \left(1 + 2 \frac{R_2}{R_1}\right). \]

As a partial check, when \( R_2 \to \infty \) we have \( T \to \infty \).

Also, when \( R_2 \to 0 \) we have \( T \to 4 \tau \left(\frac{R_2}{R_1}\right) \).