

1 Notes on spectral domain circuit analysis

1.1 Background

We learned that we can replace time derivatives by $i\omega$ when we transform from the time domain to the frequency domain, and thus solve algebraically for the transfer function between the input and output of a circuit. Let's complete the picture to go back to the time domain.

Let's recall some definitions. The transform from the time domain to the frequency domain is given by:

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt V(t) e^{-i\omega t}. \quad (1.1)$$

and the reverse transform is given by

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{V}(\omega) e^{i\omega t}. \quad (1.2)$$

Self-consistency leads to the definition of the delta function - a pulse of area one whose width goes to zero as its height goes to infinity. We have

$$\begin{aligned} \tilde{V}(\omega) &= \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \tilde{V}(\omega') e^{i\omega' t} e^{-i\omega t} \\ &= \int_{-\infty}^{\infty} d\omega' \tilde{V}(\omega') \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i(\omega-\omega')t} \end{aligned} \quad (1.3)$$

which leads to the definition of the delta function, denoted $\delta(\omega - \omega')$, as

$$\delta(\omega - \omega') \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i(\omega-\omega')t}. \quad (1.4)$$

Physically, the complex exponent oscillates between plus one and minus one so that the integral goes to zero, except, when $\omega = \omega'$, for which the integral goes to infinity. Now, on to circuits!

We previously examined the response for an RC circuit, for which the homogeneous response, defined as $\Phi(t)$, is just

$$\Phi(t) = e^{-t/\tau} \text{ for } t \geq 0 \text{ and } \Phi(t) = 0 \text{ otherwise} \quad (1.5)$$

with $\tau = RC$. For the case of the RC circuit, the complete response to an arbitrary input $V_{in}(t)$ is given by the convolution

$$V_{out}(t) = \int_0^t dx e^{-(t-x)/\tau} V_{in}(x) \quad (1.6)$$

This may be generalized, in the sense of a complicated homogeneous response, to

$$V_{out}(t) = \int_0^t dx \Phi(t-x) V_{in}(x) \quad (1.7)$$

which defines the output to an arbitrary input.

How does this look in the frequency domain? That is, how do we make use of our ability to write down the transfer function between $\tilde{V}_{in}(\omega)$ and $\tilde{V}_{out}(\omega)$. We confine our analysis to the steady state, which is to say that $\tilde{V}_{in}(\omega)$ has been present forever (so that the lower limit may be replaced with $-\infty$ and that we consider times that go to infinity (so that the upper limit may be replaced by $+\infty$). Then

$$\begin{aligned} \tilde{V}_{out}(\omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \Phi(t-x) V_{in}(x) e^{-i\omega t} \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \Phi(y) V_{in}(x) e^{-i\omega(x+y)} \\ &= \int_{-\infty}^{\infty} dy e^{-i\omega y} \Phi(y) \int_{-\infty}^{\infty} dx e^{-i\omega x} V_{in}(x) \\ &= \tilde{\Phi}(\omega) \tilde{V}_{in}(\omega) \end{aligned} \quad (1.8)$$

and we see that the process of convolution in time is transformed into multiplication in the spectral representation. $\tilde{\Phi}(\omega)$ is referred to as the transfer function that for a circuit of resistors, inductors and capacitors is just a polynomial in ω divided by another polynomial in ω .

As a simple example, the transfer function for a high-pass RC filter is

$$\tilde{\Phi}(\omega) = \frac{i\omega\tau}{1 + i\omega\tau}. \quad (1.9)$$

We learned that any periodic function - an important class of waveforms - can be written as a sum over sine waves and cosine waves, so let's take $V_{in}(t) = A \sin(\omega_0 t)$. Then

$$\begin{aligned} \tilde{V}_{in}(\omega) &= A \int_{-\infty}^{\infty} dt e^{-i\omega t} \sin(\omega_0 t) \\ &= A \int_{-\infty}^{\infty} dt \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} e^{-i\omega t} \end{aligned} \quad (1.10)$$

$$\begin{aligned}
&= A \frac{\pi}{i} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i(\omega-\omega_0)t} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i(\omega+\omega_0)t} \right] \\
&= A \frac{\pi}{i} \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right].
\end{aligned}$$

When this signal is passed through an RC high-pass filter, we have

$$\begin{aligned}
\tilde{V}_{out}(\omega) &= \tilde{\Phi}(\omega) \tilde{V}_{in}(\omega) \\
&= A \frac{\pi}{i} \left[\frac{i\omega\tau}{1+i\omega\tau} \delta(\omega - \omega_0) - \frac{i\omega\tau}{1+i\omega\tau} \delta(\omega + \omega_0) \right].
\end{aligned} \tag{1.11}$$

The last step is to transform back to the time domain. We have

$$\begin{aligned}
V_{out}(t) &= \frac{1}{2\pi} A \frac{\pi}{i} \int_{-\infty}^{\infty} d\omega \left[\frac{i\omega\tau}{1+i\omega\tau} \delta(\omega - \omega_0) - \frac{i\omega\tau}{1+i\omega\tau} \delta(\omega + \omega_0) \right] e^{i\omega t} \\
&= \frac{1}{2\pi} A \frac{\pi}{i} \left[\int_{-\infty}^{\infty} d\omega \frac{i\omega\tau}{1+i\omega\tau} \delta(\omega - \omega_0) e^{i\omega t} - \int_{-\infty}^{\infty} d\omega \frac{i\omega\tau}{1+i\omega\tau} \delta(\omega + \omega_0) e^{i\omega t} \right] \\
&= \frac{1}{2\pi} A \frac{\pi}{i} \left[\frac{i\omega_0\tau}{1+i\omega_0\tau} e^{i\omega_0 t} - \frac{-i\omega_0\tau}{1-i\omega_0\tau} e^{-i\omega_0 t} \right] \\
&= A \frac{\omega_0\tau}{1+(\omega_0\tau)^2} \frac{1}{2} \left[(1-i\omega_0\tau) e^{i\omega_0 t} + (1+i\omega_0\tau) e^{-i\omega_0 t} \right] \\
&= A \frac{\omega_0\tau}{1+(\omega_0\tau)^2} \left[(-i\omega_0\tau) \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2} + \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right] \\
&= A \frac{\omega_0\tau}{1+(\omega_0\tau)^2} \left[(\omega_0\tau) \sin(\omega_0 t) + \cos(\omega_0 t) \right] \\
&= A \frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}} \left[\frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}} \sin(\omega_0 t) + \frac{1}{\sqrt{1+(\omega_0\tau)^2}} \cos(\omega_0 t) \right]
\end{aligned} \tag{1.12}$$

Recall that $\sin(\omega_0 t + \phi) = \sin(\omega_0 t)\cos(\phi) + \cos(\omega_0 t)\sin(\phi)$. Then

$$V_{out}(t) = \frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}} \sin\left[\omega_0 t + \text{atan}(1/\omega_0\tau)\right] \tag{1.13}$$