Notes on Convolution; Physics 120, David Kleinfeld, Spring 2016

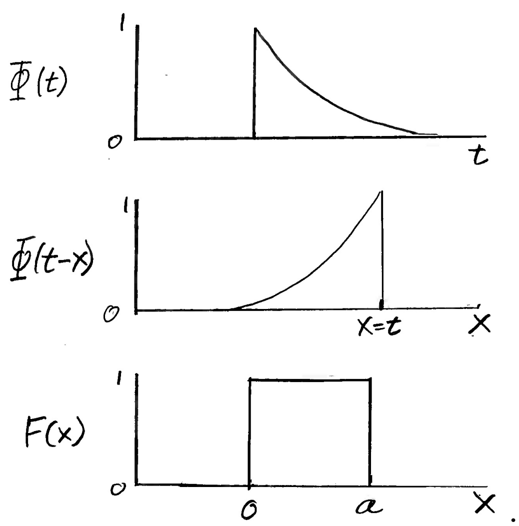
The convolution integral between the delta-function response of a circuit (the homogeneous equation), denoted Φ(t), and the external input, denoted F(t), is

$$\int_{-\infty}^{t} \mathrm{dx}\ \Phi(t-x)\ \mathrm{F}(x)$$

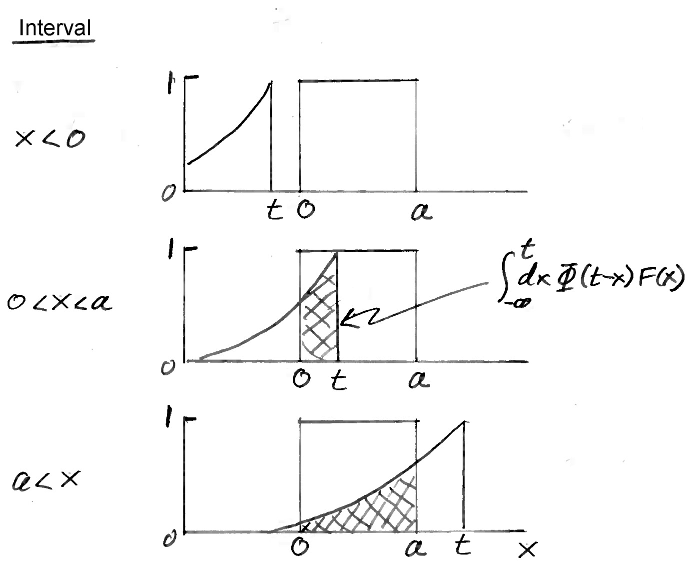
This can be seen as the overlap between Φ(-t), that is, Φ(t) running in reverse, and F(t). Here we evaluate the convolution for the choice of an exponential response

$$\Phi(t) = \frac{e^{-t/\tau}}{\tau}$$

and a pulse of height 1 and width a, as shown below:



There are three intervals of integration. In pictures:



Analytically:

For x < 0, $\int_{-\infty}^{t} \mathrm{dx}\ \Phi(t-x)\ \mathrm{F}(x) = 0$

For 0 < x < a, $\int_{-\infty}^{t} \mathrm{dx}\ \Phi(\mathrm{t}-x)\ \mathrm{F}(x) = \frac{1}{\tau}\int_{0}^{t} \mathrm{dx}\ \ e^{-(\mathrm{t}-x)/\tau} = \frac{\tau}{\tau}\ e^{-\mathrm{t}/\tau}\ \left(e^{\mathrm{t}/\tau}\ -\ 1\right) = 1 -\ e^{-\mathrm{t}/\tau}$

For a < x, $\int_{-\infty}^{t} \mathrm{dx}\ \Phi(\mathrm{t}-x)\ \mathrm{F}(x) = \frac{1}{\tau}\int_{0}^{a} \mathrm{dx}\ \ e^{-(\mathrm{t}-x)/\tau} = \frac{\tau}{\tau}\ e^{-\mathrm{t}/\tau}\ \left(e^{a/\tau}\ -\ 1\right) = e^{-(\mathrm{t}-a)/\tau}\ -\ e^{-\mathrm{t}/\tau}$

The result is an exponentially rising and decaying function that is specified by pieces; the function is continuous but the derivative is not. Pictorially: