The convolution integral between the delta-function response of a circuit (the homogeneous equation), denoted \( \Phi(t) \), and the external input, denoted \( F(t) \), is

\[
\int_{-\infty}^{t} dx \, \Phi(t-x) \, F(x)
\]

This can be seen as the overlap between \( \Phi(-t) \), that is, \( \Phi(t) \) running in reverse, and \( F(t) \).

Here we evaluate the convolution for the choice of an exponential response

\[
\Phi(t) = \frac{e^{-t/\tau}}{\tau}
\]

and a pulse of height 1 and width \( a \), as shown below:

There are three intervals of integration. In pictures:
Analytically:

For $x < 0$, \[ \int_{-\infty}^t dx \, \Phi(t-x) \, F(x) = 0 \]

For $0 < x < a$, \[ \int_{-\infty}^t dx \, \Phi(t-x) \, F(x) = \frac{1}{\tau} \int_0^t dx \, e^{-\frac{(t-x)}{\tau}} = \frac{t}{\tau} e^{-\frac{1}{\tau}} \left( e^{\frac{1}{\tau}} - 1 \right) = 1 - e^{-\frac{1}{\tau}} \]

For $a < x$, \[ \int_{-\infty}^t dx \, \Phi(t-x) \, F(x) = \frac{1}{\tau} \int_0^a dx \, e^{-\frac{(t-x)}{\tau}} = \frac{a}{\tau} e^{-\frac{1}{\tau}} \left( e^{\frac{1}{\tau}} - 1 \right) = e^{-\frac{(t-a)}{\tau}} - e^{-\frac{1}{\tau}} \]

The result is an exponentially rising and decaying function that is specified by pieces; the function is continuous but the derivative is not. Pictorially: