

# Notes on Convolution; Physics 120, David Kleinfeld, Spring 2016

The convolution integral between the delta-function response of a circuit (the homogeneous equation), denoted  $\Phi(t)$ , and the external input, denoted  $F(t)$ , is

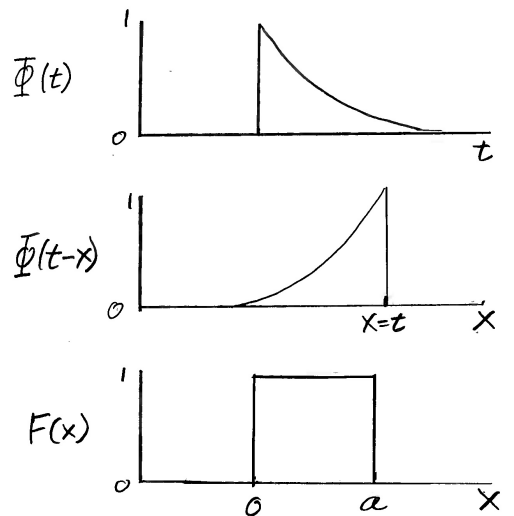
$$\int_{-\infty}^t dx \Phi(t-x) F(x)$$

This can be seen as the overlap between  $\Phi(-t)$ , that is,  $\Phi(t)$  running in reverse, and  $F(t)$ .

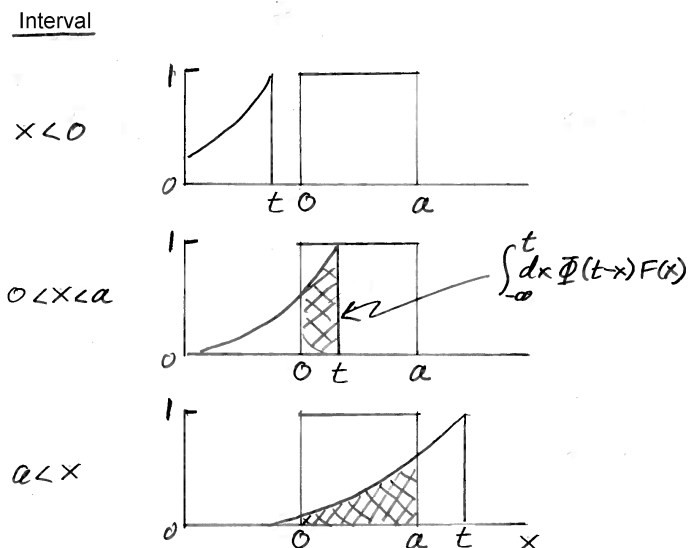
Here we evaluate the convolution for the choice of an exponential response

$$\Phi(t) = \frac{e^{-t/\tau}}{\tau}$$

and a pulse of height 1 and width  $a$ , as shown below:



There are three intervals of integration. In pictures:



Analytically:

$$\text{For } x < 0, \quad \int_{-\infty}^t dx \Phi(t-x) F(x) = 0$$

$$\text{For } 0 < x < a, \quad \int_{-\infty}^t dx \Phi(t-x) F(x) = \frac{1}{\tau} \int_0^t dx e^{-(t-x)/\tau} = \frac{\tau}{\tau} e^{-t/\tau} (e^{t/\tau} - 1) = 1 - e^{-t/\tau}$$

$$\text{For } a < x, \quad \int_{-\infty}^t dx \Phi(t-x) F(x) = \frac{1}{\tau} \int_0^a dx e^{-(t-x)/\tau} = \frac{\tau}{\tau} e^{-t/\tau} (e^{a/\tau} - 1) = e^{-(t-a)/\tau} - e^{-t/\tau}$$

The result is an exponentially rising and decaying function that is specified by pieces; the function is continuous but the derivative is not. Pictorially:

