Stability analysis of a 2-d dynamical system

January 23, 2014

The system is given by a pair of couple non-linear differential equations:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_1, x_2) \end{aligned} \tag{1}$$

A point $(x_1^\star, x_2^\star)$ is said to be a fixed point if $f(x_1^\star, x_2^\star) = g(x_1^\star, x_2^\star) = 0$. We would like to analyze the stability of the system in the neigborhood of $(x_1^\star, x_2^\star)$. To do so, we can expand the nonlinear functions about their values at the fixed point. Define $\xi_1 = x_1 - x_1^\star$ and $\xi_2 = x_2 - x_2^\star$. Using this, the Taylor series to first order are:

$$\begin{aligned} f(\xi_1, \xi_2) &\approx f(x_1^\star, x_2^\star) + \left.\frac{\partial f}{\partial x_1}\right|_{x^\star} \xi_1 + \left.\frac{\partial f}{\partial x_2}\right|_{x^\star} \xi_2 \\ &= \left.\frac{\partial f}{\partial x_1}\right|_{x^\star} \xi_1 + \left.\frac{\partial f}{\partial x_2}\right|_{x^\star} \xi_2 \end{aligned} \tag{2}$$

$$g(\xi_1, \xi_2) \approx \left.\frac{\partial g}{\partial x_1}\right|_{x^\star} \xi_1 + \left.\frac{\partial g}{\partial x_2}\right|_{x^\star} \xi_2 \tag{3}$$

where $\left.\frac{\partial f}{\partial x_1}\right|_{x^\star}$ means that the derivative is evaluated at the fixed point. Since $\dot{\xi}_1 = \dot{x}_1$ and $\dot{\xi}_2 = \dot{x}_2$ we can write the dynamics (close to the fixed point) in a compact matrix form:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}\right|_{x^\star} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{4}$$

The vector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ can be decomposed to the eigenvectors $\mathbf{v}_{1,2}$ of the matrix $\mathbf{M} = \left.\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}\right|_{x^\star}$ defined by the equation:

$$\mathbf{M}\mathbf{v}_{1,2} = \lambda_{1,2}\mathbf{v}_{1,2} \tag{5}$$

where $\lambda_{1,2}$ are the eigenvalues of the matrix $\mathbf{M}$. The solution to the linear dynamics is:

$$v_1(t) = e^{\lambda_1 t} v_1(0) \tag{6}$$

$$v_2(t) = e^{\lambda_2 t} v_2(0) \tag{7}$$

where $v_1(t)$ is the component of the vector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ along the first eigenvector of $\mathbf{M}$. We usually don't care about the full explict solution because it only applies very close to the fixed point. We do care however about the stability of the fixed point which is given by the eigenvalues $\lambda_{1,2}$.

There is a convenient formula for the eigenvalues of a 2×2 matrix:

$$\lambda_{1,2} = \frac{1}{2}\left(\mathcal{T} \pm \sqrt{\mathcal{T}^2 - 4\mathcal{D}}\right) \tag{8}$$

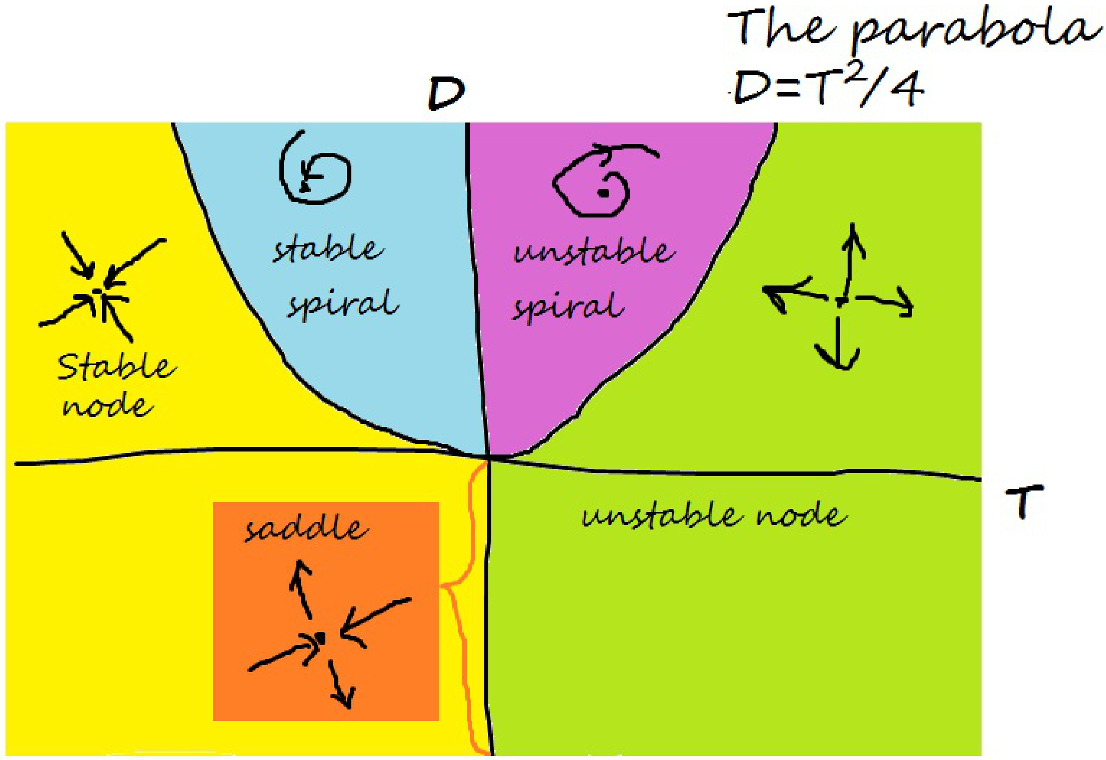

Figure 1: Stability regions in a 2-d dynamical system

where $\mathcal{T} = \text{trace}\,(\mathbf{M})$ and $\mathcal{D} = \det(\mathbf{M})$.

We can plot $\mathcal{T}$ as a function of $\mathcal{D}$ and separate the space into regions with different behaviors around the fixed point. Let's go over all the cases:

If $\mathcal{T} < 0$, both λ_1 and λ_2 have negative real parts so the fixed point is stable. If in addition $\mathcal{T}^2 \geq 4\mathcal{D}$, both eigenvalues are real so there is a "regular" stable fixed point (also called "stable node"). If $\mathcal{T}^2 < 4\mathcal{D}$ the eigenvalues are a complex conjugate pair, and the fixed point is a stable spiral (also called "stable focus").

If $\mathcal{T} > 0$, both λ_1 and λ_2 have positive real parts so the fixed point is unstable. If in addition $\mathcal{T}^2 \geq 4\mathcal{D}$, both eigenvalues are real so there is a "regular" unstable fixed point (also called "unstable node"). If $\mathcal{T}^2 < 4\mathcal{D}$ the eigenvalues are a complex conjugate pair, and the fixed point is an unstable spiral (also called "unstable focus").

If $\mathcal{T} = 0$ we separate into three cases. If $\mathcal{T}^2 > 4\mathcal{D}$, both eigenvalues are real, one of them is positive and one negative, so the fixed point is a saddle. If $\mathcal{T}^2 > 4\mathcal{D}$ the eigenvalues are purely imaginary so there are oscillations around the fixed point. The stability of the oscillations cannot be determined from linear analysis. If $\mathcal{T}^2 = 4\mathcal{D}$, linear analysis is insufficient to determine the dynamics around the fixed point.

All the cases are summarized in the plot below.

For a more detailed derivation, including explanation of the different types of bifurcations that exist when transitioning from one region to the other, refer to Eugene Izhikevich's book: Dynamical Systems in Neuroscience (chapters 3,4). The eBook is available for free through roger.ucsd.edu.