Biophysics of Neurons and Networks
Homework 1, Fall 2009

Due Tuesday October 13th
Show all code/figures/work

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Diffusion problems

1) Solving the diffusion equation
\[ \frac{\partial c}{\partial t} = -D \frac{\partial^2 c}{\partial x^2} \]
for a pulse of particles at x=0 at t=0, where c(x,t) is concentration and D is the diffusion coefficient (m^2/s),
yields the impulse response (Green’s function) for the diffusion equation
\[ G(x,t) = \frac{1}{\sqrt{4\piDt}} e^{-x^2/4Dt} \]
Plot the concentration as a function of x and t using Matlab. Also, you can use the Green’s function to derive
the time evolution of diffusion for any arbitrary initial conditions c(x,0). The way you do this is by convolving**
G(x,t) with c(x,0):
\[ c(x,t) = \int_{-\infty}^{\infty} c(\xi,0)G(x-\xi,t)d\xi \]
Do this for the initial conditions: c(x,0)=0 for x<0, c(x,0)=C_0 for x>0, and plot what you get.
You can write the answer in terms of the error function if you like.

2) The impulse response in (1) is also a probability density function for the position of a single particle starting
at x=0 at t=0. Express the standard deviation in x as a function of time t.
This problem may be solved in two ways.
\[ \sigma_x = \sqrt{<x^2> - <x>^2} = \sqrt{<x^2> - \int_{-\infty}^{\infty} x^2 p(x)dx} \]

3) Matlab: Simulate a population of random walkers that start at x=0 and move 1 step either to the left or
right at each time step. Plot the standard deviation of the positions of your population as a function of the
number of time steps elapsed (n). Assuming that \[ \sigma = n^{\frac{1}{2}} \] what is the value of beta?

4) Glutamate diffuses (D = 5 \times 10^{-6} cm^2/s) across the synaptic cleft (~20nm) in order to bind to receptors
on the postsynaptic membrane. Using the 1-dimensional impulse response, how long will it take for 50% of the
particles to reach the postsynaptic membrane.

5) We are considering diffusion of a substance m down an axon of length L cm, where m is being consumed
by some reaction at a constant rate alpha (mol/s per unit length) all along the axon. The continuity equation is
\[
\frac{\partial c_m}{\partial t} = -\frac{\partial I_m}{\partial x} - \frac{\alpha_m}{A}
\]
(the continuity equation for diffusion is just the same without the alpha term. Do you understand why it is
ture?)
where A is the constant cross sectional area of the process, and \( I_m \) is the flux of m in moles/s.
a: Combine the continuity equation with Fick’s 1st law: \( \phi = -D \frac{\partial c}{\partial x} \) to derive a modified form of the diffusion equation.

b: Show that in the steady state \( \frac{\partial c_m}{\partial t} = \frac{\partial \phi_m}{\partial x} = 0 \),
\[ c_m(x) = \frac{\alpha_m}{2DA} x^2 + a_0 x + b_0 \] is a solution and find values of the constants \( a_0 \) and \( b_0 \) corresponding to the boundary conditions \( c_m(0) = C_0 \) and \( \phi_m(l) = 0 \).

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Electrodiffusion / Nernst Problems

1) The Nernst-Planck equation relates the current density \( J \) to the flux due to diffusion
\[
\phi_{\text{diffusion}} = -D \frac{\partial c}{\partial x}
\]
to the flux due to drift of particles in an electric field
\[
\phi_{\text{drift}} = -uzF c_m(x,t) \frac{\partial \psi}{\partial x},
\]
u is the mobility (velocity given a force) of the ion
\( z \) is the valence of the ion
\( F \) is Faraday’s constant
\( \psi \) is the electrical potential

With a little algebra, we get that the current density (A/cm^2) is
\[
J(x,t) = -uzF c(x,t) \frac{\partial \ln \left( c(x,t) + zF \psi(x,t) \right)}{\partial x}
\]
Show that at steady state \( (J=0, \text{ nothing is a function of time}) \), we get the familiar
\[
\psi(x) - \psi(x_0) = \frac{RT}{zF} \ln \left( \frac{c(x_0)}{c(x)} \right)
\]

2) In 1942, Curtis and Cole studied how the membrane potential varies with the extracellular concentration of potassium (sign of the membrane potential is reversed in the figure below).

![Graph](image)

**Fig. 2** Resting potential in millivolts vs. potassium concentration of the surrounding fluid. The concentration scale is in multiples of the potassium concentration of sea-water, 13 millimolar, and is logarithmic. At high potassium concentrations the curve is a straight line, the slope of which is nearly that of the potassium electrode. In the physiological range of concentrations the potential is nearly independent of the concentration.

Using the Goldman-Hodgkin-Katz voltage equation
Explain the observation that the relationship between extracellular K+ concentration and resting membrane potential becomes increasingly linear at high extracellular [K+]. What should the slope of this relationship be at high extracellular potassium concentrations (in mV/10-fold increase in extracellular [K+])? Why might it be different in an experimental context?

3) How much does the sodium concentration change during an action potential?

\[
V_{rest} = \frac{RT}{F} \ln \frac{P_K[K^+]_{out} + P_{Na}[Na^+]_{out} + P_{Cl}[Cl^-]_{in}}{P_K[K^+]_{in} + P_{Na}[Na^+]_{in} + P_{Cl}[Cl^-]_{out}}
\]

* Contact if you are unfamiliar with Matlab, math used here, etc. We can setup a tutorial or something if enough people have similar concerns.

** Unfamiliar with convolution? [http://www.jhu.edu/signals/convolve/](http://www.jhu.edu/signals/convolve/) is intuitive. You should also know that this method of convolving the impulse response with initial conditions works for linear partial differential equations in general.