Notes on Fourier Transform.

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Let us review some math:

$$V(t) \xrightarrow[\text{Fourier}]{} \tilde{V}(\omega)$$

$$\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \; V(t)\, e^{i\omega t}$$

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \; \tilde{V}(\omega)\, e^{-i\omega t}$$

$$'' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dx \; V(x)\, e^{i\omega(x-t)}$$

$$'' = \int_{-\infty}^{+\infty} dx \; V(x) \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{i\omega(x-t)} \right]}_{\delta(x-t)}$$

Let us look @ correlation

$$C(\tau) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \; V(t)\, W(t-\tau)$$

$$\tilde{C}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\tau \; C(\tau)\, e^{i\omega\tau}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dt \; V(t)\, W(t-\tau)\, e^{i\omega\tau}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dt\, V(t)\, W(t-\tau)\, e^{-i\omega(t-\tau)}\, e^{+i\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, V(t)\, e^{+i\omega t} \int_{-\infty}^{+\infty} d\tau\, W(t-\tau)\, e^{-i\omega(t-\tau)}$$

$x = t - \tau$
$dx = -d\tau$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, V(t)\, e^{i\omega t}}_{\tilde{V}(\omega)} \; \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, W(x)\, e^{-i\omega x}}_{\tilde{W}^{*}(\omega)}$$

$$= \tilde{V}(\omega)\; W^{*}(\omega)$$

last - Conservation of energy

$$\int_{-\infty}^{+\infty} dt\, V^2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega\, \tilde{V}(\omega)\, e^{-i\omega t} \int_{-\infty}^{+\infty} dx\, \tilde{V}(x)\, e^{-ixt}$$

$$= \int_{-\infty}^{+\infty} dx\, V(x) \int_{-\infty}^{+\infty} d\omega\, \tilde{V}(\omega)\; \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{-i(\omega+x)t}}_{\delta(\omega+x)}$$

$$= \int_{-\infty}^{+\infty} d\omega\, \tilde{V}(\omega)\, \tilde{V}(-\omega)$$

but $\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty} dt\, V(t)\, e^{i\omega t}$ $\quad \tilde{V}(-\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, V(t)\, e^{-i\omega t}$

$$\therefore \int_{-\infty}^{+\infty} dt\, V^2(t) = \int_{-\infty}^{+\infty} d\omega\, \tilde{V}(\omega)\, \tilde{V}^*(\omega)$$

$$= \int_{-\infty}^{+\infty} d\omega\, |\tilde{V}(\omega)|^2$$

Power spectrum

Energy is conserved as integrated power is the same in either coordinate system

Transform pairs.

$$V(t) \equiv \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-t^2/2\sigma^2}$$

$$\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} dt\, \underbrace{e^{+i\omega t}\, e^{-t^2/2\sigma^2}}_{e^{-\frac{1}{2\sigma^2}\left(t^2 - 2\sigma^2 i\omega t - \omega^2\sigma^4\right)} \times e^{-\frac{\omega^2\sigma^2}{2}}}$$

$$'' = \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2/2(1/\sigma)^2}\, \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} dt\, e^{-\frac{1}{2\sigma^2}\left(t - i\omega\sigma^2\right)^2}}_{1}$$

Variance $= 1/\sigma^2$

Look @ convolution (~~as~~ $t \to \infty$.

$$Y(t) = \int_{-\infty}^{t} dx\, R(t-x)\, S(x)$$

$$\tilde{Y}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_{-\infty}^{t} dx\, R(t-x)\, S(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} dx\, S(x) \int_{-\infty}^{+\infty} dt\, e^{i\omega t} R(t-x)$$

$y = t-x$
$t = y+x$
$dy = dt$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} dx\, S(x)\, e^{i\omega x}}_{\substack{t \to \infty \\ \tilde{S}(\omega)}} \underbrace{\int_{-\infty}^{+\infty} dy\, e^{i\omega y} R(y)}_{\tilde{R}(\omega)}$$

$$\tilde{Y}(\omega) = \tilde{S}(\omega)\, \tilde{R}(\omega)$$