Problem #1

a) The steady-state solution is

$$r(\phi) = [W_0 r_0 + W_1 r_1 \cos\phi - \Theta]_+$$

where $[\,]_+$ indicates the Θ function.

Generally, from the Fourier series we know that

$$r_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\phi)\, e^{in\phi}\, d\phi$$

In this case

$$r_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (W_0 r_0 + W_1 r_1 \cos\phi - \Theta)\, d\phi$$

$$r_0 = W_0 r_0 - \Theta + W_1 r_1 \int_{-\pi}^{\pi} \cancel{\cos\phi}\, d\phi \quad \to 0$$

$$\Rightarrow \quad r_0 = \frac{\Theta}{W_0 - 1}$$

Similarly

$$r_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (W_0 r_0 + W_1 r_1 \cos\phi - \Theta)\cos\phi\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} W_1 r_1 \cos^2\phi\, d\phi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \cancel{(W_0 r_0 - \Theta)\cos\phi\, d\phi} \to 0$$

$$= \frac{W_1 r_1}{2}$$

$$r_1\left(1 - \frac{W_1}{2}\right) = 0 \quad \Rightarrow \quad r_1 = 0 \quad \text{or} \quad W_1 = 2$$

Substituting into the s-state solution:

$$r(\phi) = [W_0 r_0 + W_1 r_1 \cos\phi - \Theta]_+$$

$$\frac{w_0}{w_0-1}\theta - \theta \geq 0$$

$$\left(\frac{w_0 - w_0 + 1}{w_0-1}\right)\theta \geq 0$$

$\Rightarrow \quad w_0 \leq 1$ Note that there is no strong constraint on w_1 other than $w_1 \geq 0$

b) Bump state

$$r(\phi) = [w_0 r_0 + w_1 r_1 \cos\phi_c - \theta]_+$$

$$\begin{cases} \neq 0 & \text{for } -\phi_c < \phi < \phi_c \\ 0 & \text{otherwise} \end{cases}$$

In the bump state $w_0 r_0 - \theta = -w_1 r_1 \cos\phi_c$

$$r_0 = \frac{1}{2\pi}\int_{-\phi_c}^{\phi_c} (w_0 r_0 - \theta + w_1 r_1 \cos\phi)\, d\phi$$

$$= \frac{w_0 r_0 - \theta}{2\pi} 2\phi_c + \frac{w_1 r_1}{\pi}\sin\phi_c$$

$$= -\frac{w_1 r_1 \cos\phi_c}{\pi}\phi_c + \frac{w_1 r_1}{\pi}\sin\phi_c$$

$$= \frac{w_1 r_1}{\pi}(-\phi_c \cos\phi_c + \sin\phi_c)$$

$$r_1 = \frac{1}{2\pi}\int_{-\phi_c}^{\phi_c} (w_0 r_0 - \theta + w_1 r_1 \cos\phi)\cos\phi\, d\phi$$

$$= \frac{w_1 r_1}{2\pi} \int_{-\phi_c} \cos^2\phi - \cos\phi_c \cos\phi \quad d\phi$$

$$= \frac{w_1 r_1}{2\pi} \left[\phi_c - \frac{\sin 2\phi_c}{2} \right]$$

$$r_1 \left[1 - \frac{w_1}{2\pi} \left(\phi_c - \frac{\sin 2\phi_c}{2} \right) \right] = 0$$

Since $r_1 \neq 0$ then $w_1 = \dfrac{2\pi}{\phi_c - \dfrac{\sin 2\phi_c}{2}}$

Substituting into r_0

$$\frac{r_0}{r_1} = 2 \frac{(\sin\phi_c - \phi_c \cos\phi_c)}{\phi_c - \frac{\sin 2\phi_c}{2}}$$

$$w_0 = \frac{\theta}{r_0} - w_1 \frac{r_1}{r_0} \cos\phi_c$$

$$= \frac{\theta}{r_0} - \frac{\pi \cos\phi_c}{\sin\phi_c - \phi_c \cos\phi_c}$$

$$\Rightarrow \quad w_0 < \frac{-\pi \cos\phi_c}{\sin\phi_c - \phi \cos\phi_c}$$

and $$w_1 = \frac{2\pi}{\phi_c - \frac{\sin 2\phi_c}{2}}$$

2) Stability

- homogeneous state.

$$\frac{w_0}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi))\, d\phi = w_0$$

$$\frac{w_1}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi))\cos\phi\, d\phi = 0$$

$$\frac{w_1}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi))\cos^2\phi\, d\phi = 2w_1$$

the Jacobian is then

$$J = \begin{pmatrix} w_0 - 1 & 0 \\ 0 & \frac{w_1}{2} - 1 \end{pmatrix}$$

for stability, need $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$

$\Rightarrow$ $w_0 \leq 1$ and $w_1 \leq 2$

- Bump state

$$\frac{w_1}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi)) \cos\phi \, d\phi = \frac{w_1 \sin\phi_c}{\pi}$$

$$\frac{w_0}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi)) \cos\phi \, d\phi = \frac{w_0 \sin\phi_c}{\pi}$$

$$\frac{w_1}{2\pi}\int_{-\pi}^{\pi} \Theta(r(\phi)) \cos^2\phi \, d\phi = \frac{w_1}{2\pi}\left(\phi_c + \sin\phi_c \cos\phi_c\right)$$

the Jacobian is then

$$J = \begin{pmatrix} \frac{w_0 \phi_c}{\pi} - 1 & \frac{w_1 \sin\phi_c}{\pi} \\ \frac{w_0 \sin\phi_c}{\pi} & \frac{w_1}{2\pi}\left(\phi_c + \underbrace{\sin\phi_c \cos\phi_c}_{\phi_c - \frac{2\pi}{w_1}}\right) - 1 \end{pmatrix}$$

stability if $\operatorname{tr} J < 0$ and $\det J > 0$

$$\operatorname{tr} J = \frac{1}{\pi}\left(w_0 \phi_c - \pi + w_1 \phi_c - 2\pi\right)$$

$$= \frac{1}{\pi}\left(w_0 \phi_c + w_1 \phi_c - 3\pi\right) < 0$$

$$\Leftrightarrow \quad w_0 + w_1 < \frac{3\pi}{\phi_c}$$

$$\det J = \frac{1}{\pi^2}\left((w_0 \phi_c - \pi) w_1 \sin\phi_c \cos\phi_c - w_0 w_1 \sin^2\phi_c\right)$$

$$= \frac{w_1}{\pi^2} \sin\phi_c \left(w_0 \left[\phi_c \cos\phi_c - \sin\phi_c\right] - \pi \cos(\phi_c)\right).$$

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$$\det J > 0 \iff w_0 (\phi_c \cos\phi_c - \sin\phi_c) > \pi \cos\phi_c$$

$$\cos\phi_c + \frac{w_0}{\pi}(\sin\phi_c - \phi_c \cos\phi_c) < 0$$

$$\iff \quad w_0 < \frac{\pi}{\phi_c(w_1) - \tan(\phi_c(w_1))}$$

where $\quad w_1(\phi_c)) = \dfrac{2\pi}{\phi_c - \sin\phi_c \cos\phi_c}$