Winter 2019 PHYS 178/278, Homework 2
Due 11:59 PM on February 10^{th}

Please submit your assignment as one "LastName, FirstName_PID_HW2.pdf" file via email (whsu@physics.ucsd.edu).
Welcome to use any software or programming language you are familiar for doing the simulation.

1 Two Dimensional Neuron[1]

In this problem we will study the dynamics of a neuron described by a membrane potential variable V and a recovery variable W. The dynamics of the neuron are in general:

$$\frac{d}{dt}\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \dot{V} \\ \dot{W} \end{pmatrix} = \begin{pmatrix} F_1(V,W) \\ F_2(V,W) \end{pmatrix}. \tag{1}$$

Assume that for all values of V and W:

$$\frac{\partial F_1(V,W)}{\partial W} < 0, \qquad \frac{\partial F_2(V,W)}{\partial W} < 0, \qquad \frac{\partial F_2(V,W)}{\partial V} > 0.$$

An equilibrium point (V_0, W_0) is the point where $\dot{V} = F_1(V_0, W_0) = 0$ and $\dot{W} = F_2(V_0, W_0) = 0$.

1.1 Suppose that at the equilibrium point $\frac{\partial F_1}{\partial V} < 0$. **Show that the equilibrium is stable.**

Nullclines are the lines where either $\dot{V}$ or $\dot{W}$ are 0. If the nullclines intersect, that is an equilibrium point. We will call the $F_1 = 0$ line the "V nullcline" (because $\dot{V} = 0$ on that line) and the $F_2 = 0$ line the "W nullcline".

A specific model

We will work with Fitzhugh-Nagumo model of the form:

$$F_1(V,W) = f(V) - W + I, \text{ where } f(V) = V - \frac{1}{3}V^3 \tag{2}$$

$$F_2(V,W) = \phi(V - bW) \tag{3}$$

1.2 Plot the nullclines of the model (solve for $W(V)$ such that F_1 or F_2 are 0, see Fig. 1 for the the sketch) with the parameters $I = 3$, $b = 1/2$. Note that the V nullcline has three "branches" and the W nullcline is monotonic.

1.3 Find parameters such that the equilibrium is in the middle branch of the V nullcline. Compute $\frac{\partial F_1}{\partial V}$ at the equilibrium and show that it is positive. Given your answer to **1.1,** what does that say about the equilibrium point?

1.4 Now find two sets of parameters and plot the nullclines for each of them:

(a) One such that **the equilibrium is in the middle branch of the V nullcline** and the **slope** of the V nullcline is **smaller** than the slope of the W nullcline at the equilibrium point.

(b) Another such that **the equilibrium is in the middle branch of the V nullcline** and the **slope** of the V nullcline is **greater** than the slope of the W nullcline at the equilibrium point.

[1]Problem courtesy of Bard Ermentrout. A good resource that can help with some background is chapter 3 of Bard's book: Mathematical Foundations of Neuroscience. The eBook is available for free through http://roger.ucsd.edu/

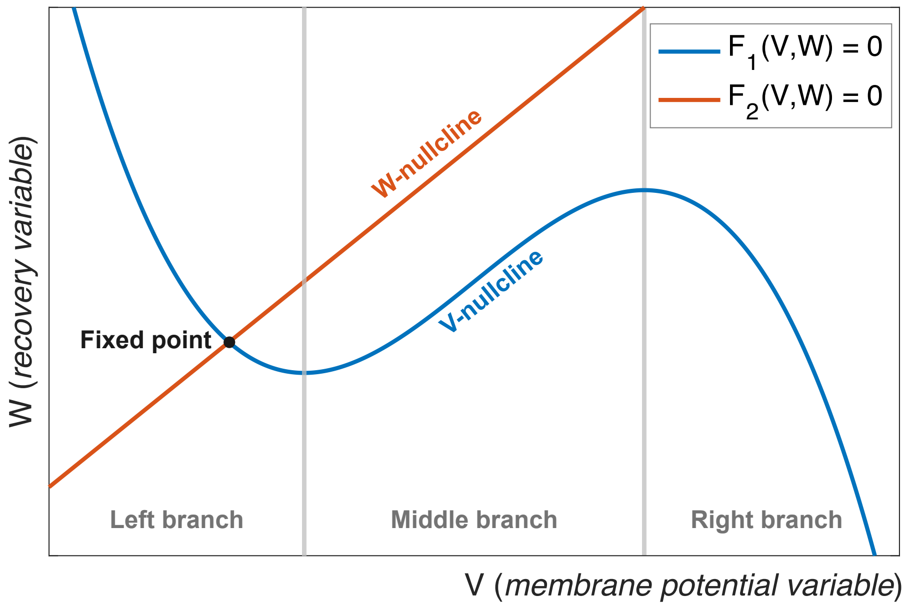

Figure 1: Sketch of V and W nullclines.

Use linear stability analysis to show that in the first case the equilibrium is a node and that in the second case it is a saddle. Can you use the nullcline plots to explain this graphically?

1.5 Run the dynamics. Plot examples of spike trains and discuss the different regimes you identified above.

2 Noise and "Balanced" Network

Assume that a neuron in a neural network receives both K excitatory and K inhibitory inputs from the presynaptic neurons, each of which spikes ($S_j = 1$) with probability m and silences ($S_j = 0$) with probability $1 - m$. The total input sending to the postsynaptic neuron is

$$\mu = \sum_{j=1}^{K} W_j^E S_j^E + \sum_{J=1}^{K} W_j^I S_j^I,$$

where the weights, $W_j^E(K)$ and $W_j^I(K)$, depend on the number of connections K. We would like to explore the relation between **the mean** $\langle\mu\rangle$, **the variance Var** (μ) **of total inputs** and **the number of presynaptic connections** (K) with numerical simulations, given two scenarios:
(1) A postsynaptic neuron receives purely excitatory inputs from K presynaptic neurons ($W_j^E \neq 0$, $W_j^I = 0$);
(2) A postsynaptic neuron receives both K excitatory and K inhibitory inputs ($W_j^{E,I} \neq 0$).

2.1 Consider the case of purely K **excitatory inputs** ($W_j^E = \frac{1}{K}$, $W_j^I = 0$),

$$\mu = \sum_{j=1}^{K} W_j^E S_j^E = \frac{1}{K} \sum_{j=1}^{K} S_j^E.$$

To calculate the mean $\langle\mu\rangle$ and the variance $\mathrm{Var}(\mu)$ of the total input μ, firstly we need to generate K excitatory inputs for one time step, and extend it to a sequence (i.e., multiple time steps) over the time duration T. Then, we can calculate the average $\langle\mu\rangle$ and the variance $\mathrm{Var}(\mu)$ from this input sequence. Here are the steps for you to follow:

STEP 1: Generate K excitatory inputs occurring in one time step (dt) as an one-dimensional binary vector

$$\boldsymbol{S}^E = \begin{pmatrix} S_1^E \\ S_2^E \\ \vdots \\ \vdots \\ S_K^E \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Hint: use MATLAB command ($\texttt{rand(K,1)} <= \texttt{m} * \texttt{dt}$) to generate a K-by-1 column vector, where $\texttt{m} * \texttt{dt}$ is the spiking probability of one time bin (dt). You can choose the spiking probability (m) between $0 < m \leq 1$.

STEP 2: Extend the K excitatory inputs (K-by-1) from one-time step to a sequence over a period of time $T \gg dt$ (ex: you may set $dt = 1$ and $T = 10000$). *Hint*: how many dt bins (`n`) are there in the time interval T? Find **the number of bins (`n`)** and use ($\texttt{rand(K,n)} <= \texttt{m} * \texttt{dt}$) to generate a K-by-n binary matrix. Each column represents K inputs at each time step, while each row is the input sequence sending from one of the presynaptic neurons.

$$\boldsymbol{S}^E\{t\} = \begin{array}{c} 1^{\text{st}} \\ 2^{\text{nd}} \\ \vdots \\ K^{\text{th}} \end{array} \begin{pmatrix} | & | & | & & | \\ \boldsymbol{S}^E(0) & \boldsymbol{S}^E(dt) & \boldsymbol{S}^E(2dt) & \cdots & \boldsymbol{S}^E(T) \\ | & | & | & & | \end{pmatrix}_{K \text{ by } n}$$

Time stpes from 0 to T

STEP 3: From the K-by-n matrix, how would you get a total input μ received at each time step?
Hint: $\texttt{(1/K)} * \texttt{sum(rand(K,n)} <= \texttt{m} * \texttt{dt,1)}$ returns the sum of each column, i.e., an 1-by-n row vector.

STEP 4: Now you have input sequence μ (1-by-n row vector), you will be able to calculate **the mean** $\langle\mu\rangle$ and **the variance Var** (μ). *Hint*: Try to look up some built-in MATLAB functions for computing mean and variance.

STEP 5: Write a for-loop that calculate the mean $\langle\mu\rangle$ and variance $\text{Var}(\mu)$ with different connections $K = 200$, 400, 600, 800, 1000, 1500, 2000, 3500, and 5000. For each K, you may want to run multiple trials and take the average.

STEP 6: Plot $\langle\mu\rangle$ **vs** K, and **Var**(μ) **vs** K. Compare each curve with zero (horizontal line $y = 0$, label it on the plot would be helpful), and briefly describe the trend for each.

2.2 K excitatory and K inhibitory inputs ($W_j^E = -W_j^I = \frac{1}{K}$),

$$\mu = \frac{1}{K}\left(\sum_{j=1}^{K} S_j^E - \sum_{j=1}^{K} S_j^I\right).$$

Extend the simulation steps in **2.1**, plot $\langle\mu\rangle$ vs K and $\text{Var}(\mu)$ vs K for a case that a neuron receives both K excitatory and K inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), and briefly describe the trend for each.

2.3 K excitatory and K inhibitory inputs, with the weights $W_j^E = -W_j^I = \frac{1}{\sqrt{K}}$,

$$\mu = \frac{1}{\sqrt{K}}\left(\sum_{j=1}^{K} S_j^E - \sum_{j=1}^{K} S_j^I\right).$$

Extend the simulation steps in **2.1**, plot $\langle\mu\rangle$ vs K and $\text{Var}(\mu)$ vs K for a case that a neuron receives both K excitatory and K inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), briefly describe the trend for each, and compare them with the results you got from **2.1** and **2.2**.