# HW2 Prob. 1 Clarification

## 1 Two Dimensional Neuron[1]

In this problem we will study the dynamics of a neuron described by a membrane potential variable $V$ and a recovery variable $W$. The dynamics of the neuron are in general:

$$\frac{d}{dt}\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \dot{V} \\ \dot{W} \end{pmatrix} = \begin{pmatrix} F_1(V,W) \\ F_2(V,W) \end{pmatrix}. \tag{1}$$

Assume that for all values of $V$ and $W$:

$$\frac{\partial F_1(V,W)}{\partial W} < 0, \qquad \frac{\partial F_2(V,W)}{\partial W} < 0, \qquad \frac{\partial F_2(V,W)}{\partial V} > 0.$$

An equilibrium point $(V_0, W_0)$ is the point where $\dot{V} = F_1(V_0, W_0) = 0$ and $\dot{W} = F_2(V_0, W_0) = 0$.

**1.1** Suppose that at the equilibrium point $\frac{\partial F_1}{\partial V} < 0$. **Show that the equilibrium is stable.**

# HW2 Prob. 1.1

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- The linearized matrix:

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial W} \\ \frac{\partial F_2}{\partial V} & \frac{\partial F_2}{\partial W} \end{pmatrix}_{(V_0, W_0)} = \begin{pmatrix} - & - \\ + & - \end{pmatrix}$$

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$$T = \text{Trace}\,(A) = (-) + (-) < 0$$
$$D = \text{Det}\,(A) = (-)(-) - (-)(+) > 0$$

Then, we can determined the stability based on *T* and *D.*
(see Jan. 30th discussion notes)

# HW2 Prob. 1.3

## A specific model

We will work with Fitzhugh-Nagumo model of the form:

$$F_1(V,W) = f(V) - W + I, \text{ where } f(V) = V - \frac{1}{3}V^3 \quad (2)$$

$$F_2(V,W) = \phi(V - bW) \quad (3)$$

**1.3** Find parameters such that the equilibrium is in the middle branch of the $V$ nullcline. Compute $\frac{\partial F_1}{\partial V}$ at the equilibrium and show that it is positive. Given your answer to **1.1,** what does that say about the equilibrium point?

- $W$ is the slow (recovery) variable, the time constant $\tau_w = \frac{1}{\phi} \gg 1$

$$\Rightarrow \; 0 < \phi \ll 1$$

- The parameter $b$ sets the scale of growth versus decay $b > 0$
(See Week 2 notes on reduced spike model)

- Input $I \in [-\infty, \infty]$

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- The linearized matrix:

  Consider $0 < \phi \ll 1, b > 0$

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial W} \\ \frac{\partial F_2}{\partial V} & \frac{\partial F_2}{\partial W} \end{pmatrix}_{(V_0,W_0)} = \begin{pmatrix} 1 - V_0^2 & -1 \\ \phi & -b\phi \end{pmatrix} = \begin{pmatrix} ? & - \\ + & - \end{pmatrix}$$

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- The linearized matrix:

  Consider $0 < \phi \ll 1, b > 0$

It's for you to figure out!

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- The linearized matrix:

After you found out $\frac{\partial F_1}{\partial V} > 0$

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial W} \\ \frac{\partial F_2}{\partial V} & \frac{\partial F_2}{\partial W} \end{pmatrix}_{(V_0, W_0)} = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

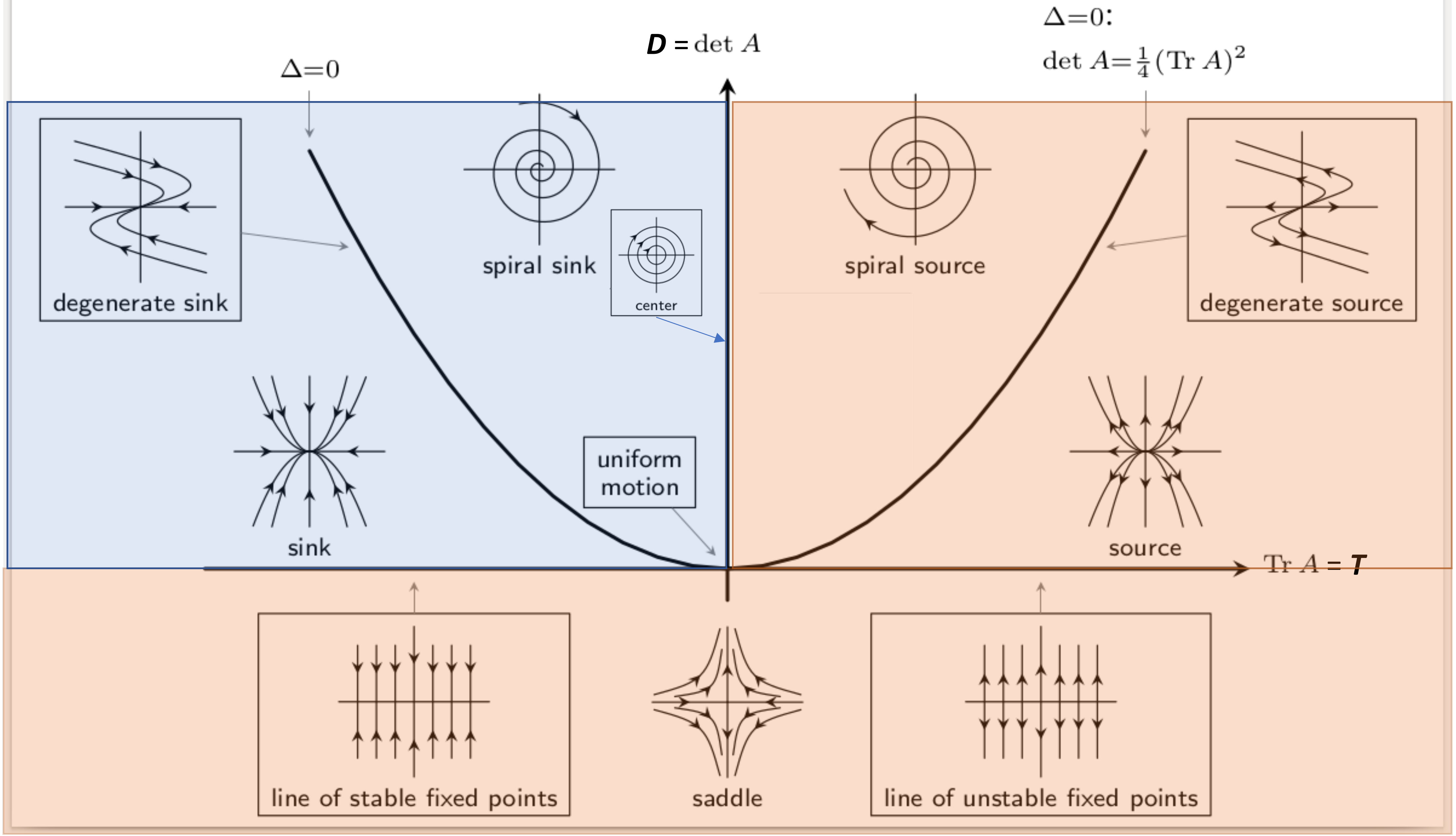
$$T = \text{Trace}(A) = (+) + (-) = ?$$

$$D = \text{Det}(A) = (+)(-) - (-)(+)$$
$$= (-) + (+) = ?$$

**We don't know the sign of *T* and *D***, so cannot determine the stability as easily as Prob. 1.1

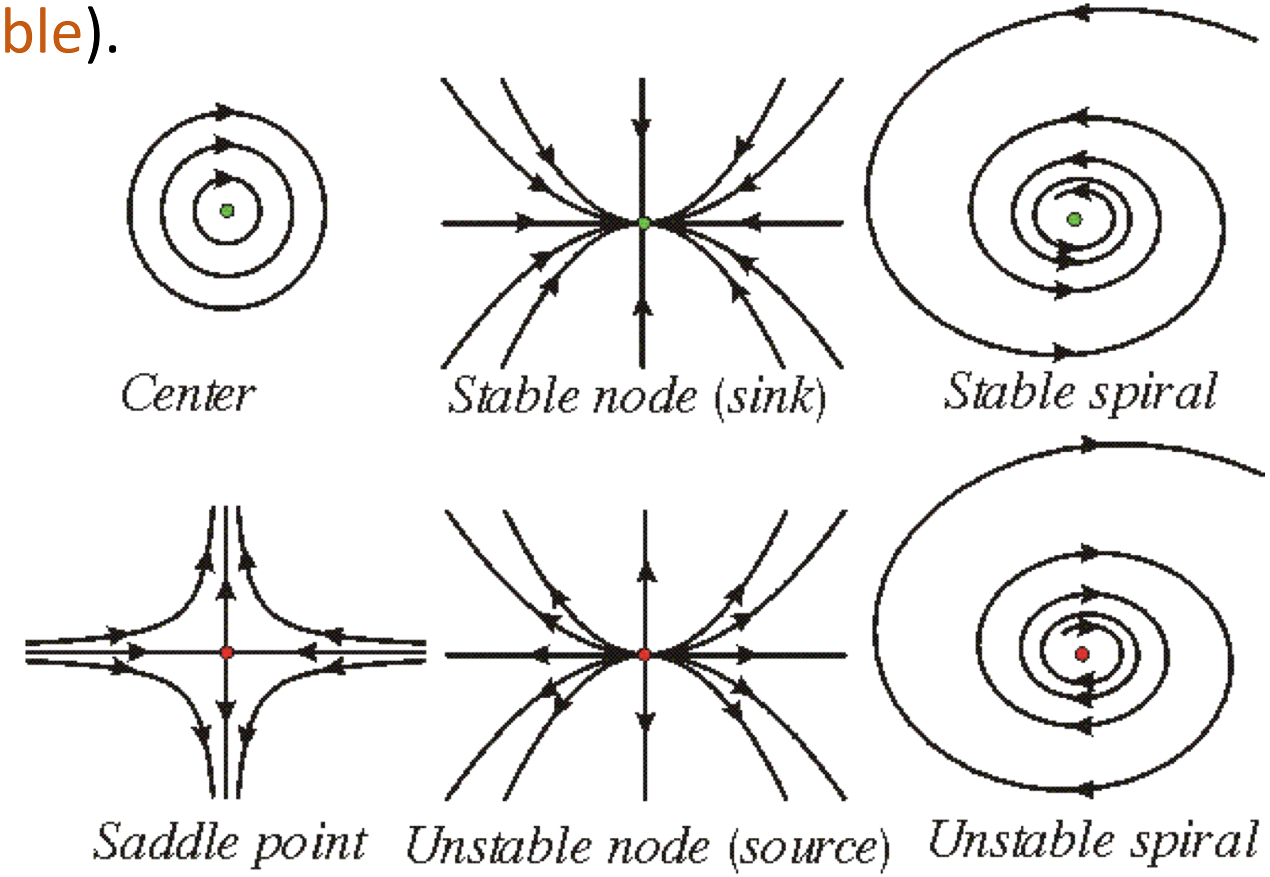
→ It's your turn to analyze the stability with different values of $b$ and $\phi$ ( Prob. 1. 4 (a) (b) ).

# Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane



# Classification of Linear Stability (2-D system)

**Center** ($D > 0$, $T = 0$ ).
**Node** ($0 < D < T^2/4$ ), ($T > 0$ unstable; $T < 0$ stable).
**Spiral** ($D > T^2/4$ ), ($T > 0$ unstable; $T < 0$ stable).
**Saddle** ($D < 0$, unstable).




http://staff.www.ltu.se/~larserik/applmath/chap9en/part7.html