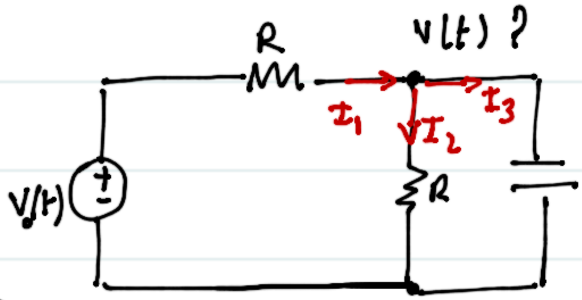
Phys 178/278 Homework #1 solutions

Question #1 - Circuits

a)



$$V_0(t) = \begin{cases} 0 & t<0 \\ V_0 & t \geq 0 \end{cases}$$

$$V(t) = \frac{Q_3}{C}$$

From Kirchoff's laws:

① $$I_1 = I_2 + I_3$$

② $$V_0(t) - I_1 R - I_2 R = 0$$

③ $$\frac{Q_3}{C} - I_2 R = 0$$

Substituting into ② $$V_0(t) - I_1 R - \frac{Q_3}{C} = 0$$

From ① $$V_0(t) - (I_2 + I_3) R - \frac{Q_3}{C} = 0$$

$$V_0(t) - 2\frac{Q_3}{C} - I_3 R = 0$$

$$I_3(t) = -\frac{2 Q_3}{RC} + \frac{V_0(t)}{R}$$

need to solve

$$\frac{dQ_3}{dt} = -\frac{2Q_3}{RC} + \frac{V_0(t)}{R}$$

$$Q_3(t) = \frac{V_0 C}{2}\left(1 - e^{-\frac{2}{RC}t}\right)$$

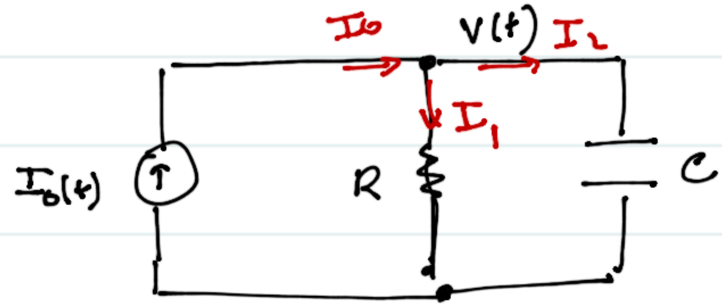
$$\Rightarrow \quad V(t) = \frac{V_0}{2}\left(1 - e^{-2/RC\, t}\right)$$

Check whether limits make sense.

$$V(t=0) = 0 \checkmark$$

$$V(t \to \infty) = \frac{V_0}{2} \checkmark$$

b)



$$I_0(t) = \begin{cases} 0 & t < 0 \\ I_0 & 0 \leq t < a \\ 0 & t \geq a \end{cases}$$

For $t < 0$ $\quad I_0 = 0$

$$V(t) = I_0 R = 0$$

For $0 \leq t < a$

① $I_0 = I_1 + I_2$

② $I_1 R - \dfrac{Q_2}{C} = 0$

② → ① $\quad I_0 = \dfrac{Q_2}{RC} + I_2$

$$\frac{dQ_2}{dt} = I_0 - \frac{Q_2}{RC}$$

$$Q_2(t) = I_0 RC\left(1 - e^{-t/RC}\right)$$

$$V(t) = I_0 R\left(1 - e^{-t/RC}\right)$$

For $t \geq a$

(1) $I_1 = -I_2$

(2) $I_1 R - \frac{Q_2}{C} = 0$

$$I_2 = -\frac{Q_2}{RC}$$

$$\frac{dQ_2}{dt} = -\frac{Q_2}{RC}$$

$$Q_2(t) = Q_2(a)\, e^{-(t-a)/RC}$$

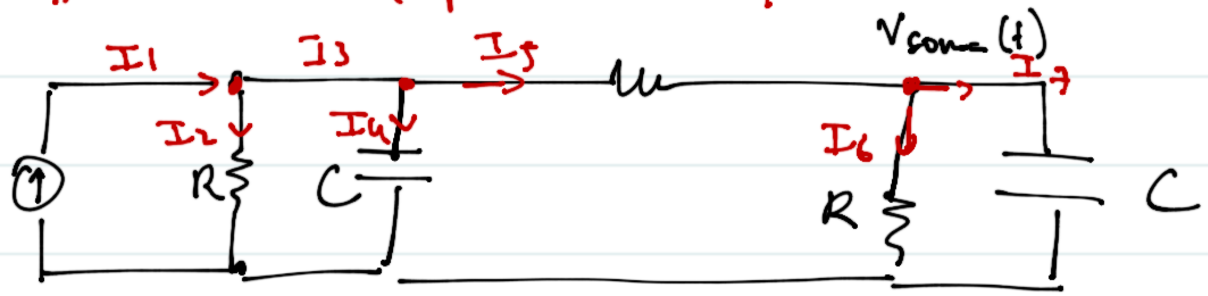
$$V(t) = \frac{Q_2(a)}{C}\, e^{-(t-a)/RC}$$

$$V(t) = I_0 R\left(1 - e^{-a/RC}\right) e^{-(t-a)/RC}$$

$$V(t) = \begin{cases} 0 & t < 0 \\ I_0 R\left(1 - e^{-t/RC}\right) & 0 \leq t < a \\ I_0 R\left(1 - e^{-a/RC}\right) e^{-(t-a)/RC} & t \geq a \end{cases}$$

$V(t)$ is continuous at $t = 0$ and $t = a$ ✓

Question # Circuit representation of a neuron



KCL:

$V_{soma} = I_6 R.$

$= \frac{Q_7}{C}$

① $I_1 = I_2 + I_3$

② $I_3 = I_4 + I_5$

③ $I_5 = I_6 + I_7$

KVL:

④ $I_{syn} R = I_2 R$

⑤ $I_2 R - \frac{Q_4}{C} = 0$

⑥ $\frac{Q_4}{C} - I_5 R + I_6 R = 0$

⑦ $I_6 R - \frac{Q_7}{C} = 0$

⑥ → ⑦ $\quad \frac{Q_u}{c} - I_5 R = \frac{Q_7}{c}$

⑤ → $\quad I_2 R - I_5 R = \frac{Q_7}{C}$.

④ → $\quad I_{syn} R - I_5 R = \frac{Q_7}{C}$

③ → $\quad I_{syn} R - (I_6 + I_7) R = \frac{Q_7}{c}$

$$I_{syn} R - I_7 R = \frac{2 Q_7}{C}$$

For $\quad t < 0$

$$\frac{dQ_7}{dt} = - \frac{2 Q_7}{RC}$$

$\tau = RC.$

$$Q_7(t) = Q_0 \, e^{-2t/\tau}$$

$$Q_7(t=0) = 0 \quad \Rightarrow \quad Q_0 = 0.$$

For $\quad 0 < t < a$

$$I_0 R - \frac{dQ_7}{dt} R = \frac{2 Q_7}{C}$$

$$\frac{dQ_7}{dt} = I_0 - \frac{2 Q_7}{RC}.$$

$$Q_7(t) = I_0 RC \left(1 - e^{-2t/RC}\right)$$

$$Q_7(t=0) = 0 \checkmark$$

$$Q_7(t=a) = I_0 RC \left(1 - e^{-2a/RC}\right)$$

For $t > a$.

$$\frac{dQ_7}{dt} = \frac{-2Q_7}{RC}$$

$$Q_7(t) = Q_7(t=a)\, e^{-2(t-a)/RC}$$

$$Q_7(t) = I_0 RC \left(1 - e^{-2a/RC}\right) e^{-2(t-a)/RC}$$

$$V_{soma}(t) = \begin{cases} 0 & t < 0 \\ I_0 R \left(1 - e^{-2t/RC}\right) & 0 \leq t < a \\ \underbrace{I_0 R \left(1 - e^{-2a/RC}\right)}_{I_{max}} e^{-2(t-a)/RC} & t \geq a \end{cases}$$

Question # 4 Two Linear neurons w/ mutual inhibition

$$S_i^{t+1} = \begin{cases} 1 & w_1 S_1^t > \theta_2 \\ -1 & w_1 S_1^t < \theta_2 \\ S_1^t & w_1 S_1^t = \theta_2 . \end{cases}$$

a) S_i^t is a fixed point $\iff$ $S_i^{t+1} = S_i^t$

- $S^t = (+1, +1)$

$$S_1^{t+1} = \mathrm{sgn}(w_2 S_2^t - \theta_1)$$
$$= \mathrm{sgn}(w_2 - \theta_1)$$

$$S_2^{t+1} = \mathrm{sgn}(w_1 S_1^t - \theta_2)$$
$$= \mathrm{sgn}(w_1 - \theta_2)$$

$$S^{t+1} = \begin{pmatrix} \mathrm{sgn}(w_2 - \theta_1) \\ \mathrm{sgn}(w_1 - \theta_2) \end{pmatrix} = \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

$\Rightarrow$ $w_2 > \theta_1$ and $w_1 > \theta_2$

- $S^t = (+1, -1)$

$$S^{t+1} = \begin{pmatrix} \mathrm{sgn}(-w_2 - \theta_1) \\ \mathrm{sgn}(+w_1 - \theta_2) \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$\Rightarrow \quad -w_2 > \theta_1 \quad$ and $\quad \theta_2 > +w_1$

- $s^t = (-1, +1)$

$$s^{t+1} = \begin{pmatrix} sgn\ (+w_2 - \theta_1) \\ sgn\ (-w_1 - \theta_2) \end{pmatrix} \doteq \begin{pmatrix} -1 \\ +1 \end{pmatrix}$$

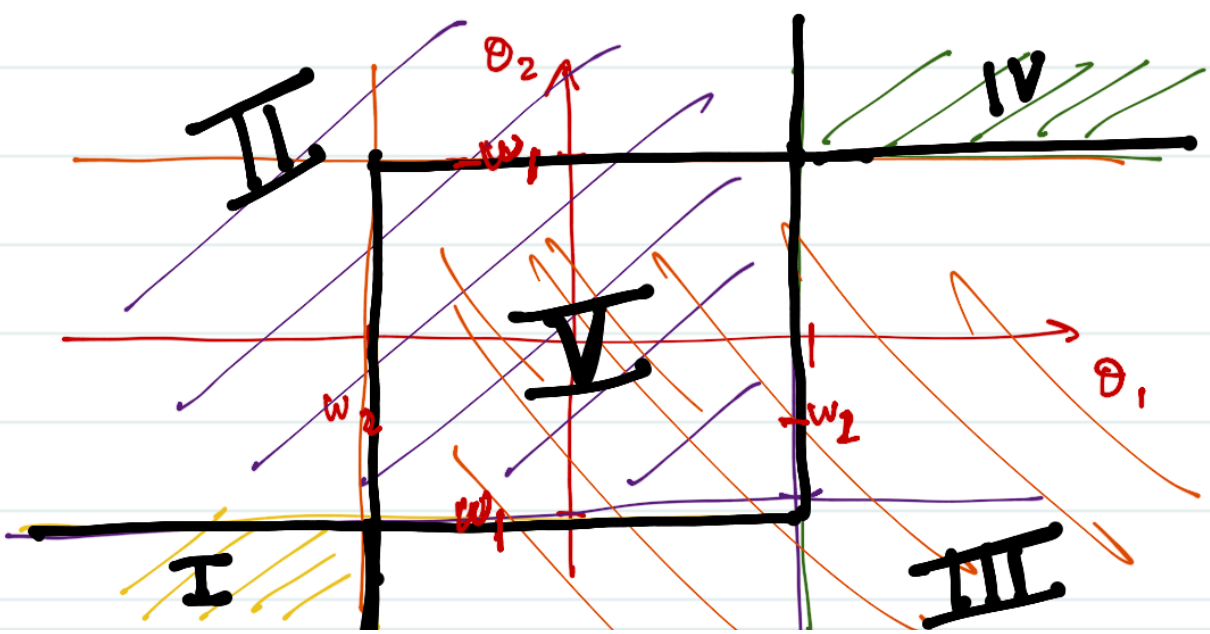
$\Rightarrow \quad \theta_1 > +w_2 \quad$ and $\quad -w_1 > \theta_2$

- $s^t = (-1, -1)$

$$s^{t+1} = \begin{pmatrix} sgn\ (-w_2 - \theta_1) \\ sgn\ (-w_1 - \theta_2) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$\Rightarrow \quad \theta_1 > -w_2 \quad$ and $\quad \theta_2 > -w_1$

2- Recall $w_1, w_2 < 0$



Question #5 Binary neurons w/ symmetric connection

1- $\Delta E = -(\mu_i - \theta_i)\,\Delta S_i$

where $\mu_i = \sum_{j=1, j\neq i} w_{ij} S_j + I_{i,ext}$

and $\Delta S_i = S_i^{t+1} - S_i^{t}$

Case 1: $\Delta E = 0 \iff \mu_i = \theta_i$ or $\Delta S_i = 0$

. if $\mu_i = \theta_i$, then $S_i^{t+1} = S_i^{t}$

so there is no change in S_i

. if $\Delta S_i = 0$, then $S_i^{t+1} = S_i^{t}$

so there is no change in S_i

therefore $\Delta E = 0 \iff$ no change in neuron state

Case 2: $\Delta E < 0 \iff (\mu_i - \theta_i)\,\Delta S_i > 0$

. $\mu_i > \theta_i$ and $\Delta S_i > 0$

if $\mu_i > \theta_i$ then $S_i^{t+1} = +1$

since $\Delta S_i = S_i^{t+1} - S_i^{t} > 0$

then $S_i^{t} = -1$

the state of the neuron flips

- $\mu_i < \theta_i$ <u>and</u> $\Delta S_i < 0$

  if $\mu_i < \theta_i$ then $S_i^{t+1} = -1$

  Since $\Delta S_i = S_i^{t+1} - S_i^t < 0$

  then $S_i^t = +1$

  the state of the neuron flips.

therefore the state of the neuron flips $\Leftrightarrow \Delta E < 0$

2. Direction 1 : Fixed point $\rightarrow$ local min.

$\vec{S}$ is a fixed point $\Leftrightarrow \Delta S = 0$, $S_i^{t+1} = S_i^t$

From part a) Case ① we've shown that $\Delta E = 0 \Leftrightarrow \Delta S = 0$

Suppose we flip the neuron $S_i^{t+1} = -S_i^t$ and get a local minim $\Delta E \geq 0$. By part 1 Case ② we've shown that $\Delta E < 0$. this case is therefore not possible.

therefore if S_i^t is a fixed point it's also a local min.

Direction 2: local min $\rightarrow$ fixed point.

A state is a local min if $\Delta E \geq 0$ $\forall_i : S_i^t \rightarrow -S_i^t$

But we've shown from part a) that if a neuron flips, $\Delta E < 0$

therefore S_i^t cannot flip for $\Delta E \geq 0$

and S_i^t is a fixed point.

3- From part a) we've shown that any update $\rightarrow$ $\Delta E < 0$
and from part b) if $\vec{S}$ is a fixed point it's also
a local minimum

$$S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \dots \rightarrow S^t$$

$$E_1 \geq E_2 \geq E_3 \quad \dots \quad \geq E^t$$

Since the number of states is finite $\exists \; E_j = \min \{E_i\}$

Since $\Delta E \leq 0$ $\quad \Delta E_j = 0$

We've shown that if $\Delta E = 0$ the states do not flip
and $\Delta S = 0$

therefore the system always reaches a fixed point at $t \to \infty$
if the number of states is finite