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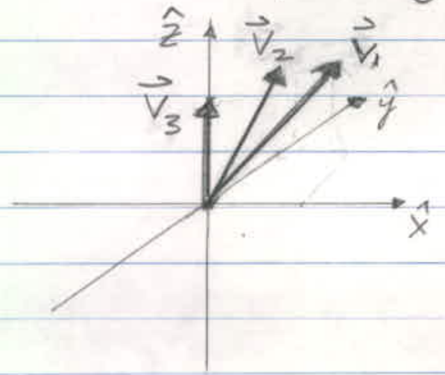
Brief tutorial on vectors & matrices
- NEUroinFOrmatics summer 2010 -

Vectors are an ordered set of numbers that define a point in $\mathbb{R}^n$

A set of k vectors with n entries can span up to n dimensions for $k \geq n$.

- Consider the spanning set $S = \{\vec{V}_1\ \vec{V}_2\ \vec{V}_3\}$;



$$" = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

↑ 3 rows by 1 column

Does this set span $\mathbb{R}^3$? If so, an arbitrary vector can be expressed in terms of $\vec{V}_1$, $\vec{V}_2$, and $\vec{V}_3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

arbitrary vector

expansion coefficients

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a+b \\ a+b+c \end{pmatrix}$$

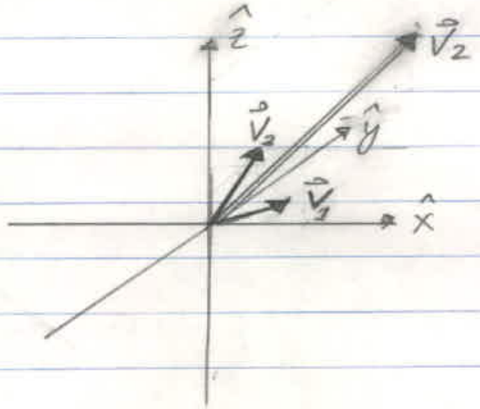
$$\begin{aligned} a &= x \\ a+b &= y \Rightarrow b = y-a = y-x \\ a+b+c &= z \Rightarrow c = z-a-b = z-\cancel{x}-y+\cancel{x} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (y-x)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (z-y)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus the three vectors span $\mathbb{R}^3$.

- Let's consider a second example:

$$S = \{\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3\} = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+2b \\ a+5b+c \\ 3b+c \end{pmatrix}$$

$$y - z = a + \cancel{5}^{2}b + \cancel{c} - 3b - \cancel{c} = a + 2b$$
$$\quad " \quad = x$$

Here S is <u>not</u> a spanning set as only the special vector $\begin{pmatrix} y-z \\ y \\ z \end{pmatrix}$, which defines a plane, is represented. All of $\mathbb{R}^3$ is not reachable. In other words, $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are not linearly independent

- Linear independence is when no vector can be represented by a sum over vectors in the set.

Dependent, for a set of k vectors, if

$$\vec{v}_j = a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} + a_{j+1}\vec{v}_{j+1} + \cdots + a_k\vec{v}_k$$

Linear dependent if number of vectors exceeds dimension of space.

Example: $\{\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{spans } \mathbb{R}^3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\vec{v}_4 = \vec{v}_1 - \vec{v}_2$

Size of linearly independent set is less than or equal to the size of the spanning set.

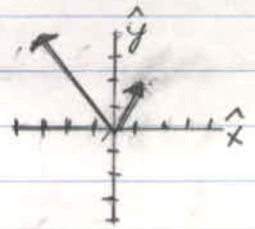
Example; $\{\vec{V}_1\ \vec{V}_2\ \vec{V}_3\} = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$

$\vec{V}_3 = \vec{V}_1 - \vec{V}_2$

this does not span $\mathbb{R}^3$.

- Orthonormal bases are vectors that span a space and have a length of one, i.e., unit vector. Let's see how to form these.

Consider $\{\vec{V}_1\ \vec{V}_2\} = \left\{ \begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}-3\\4\end{pmatrix} \right\}$



$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}-3\\4\end{pmatrix}$

$\left.\begin{matrix} x = a-3b \\ y = 2a+4b \end{matrix}\right\} 2x-y=-2b$

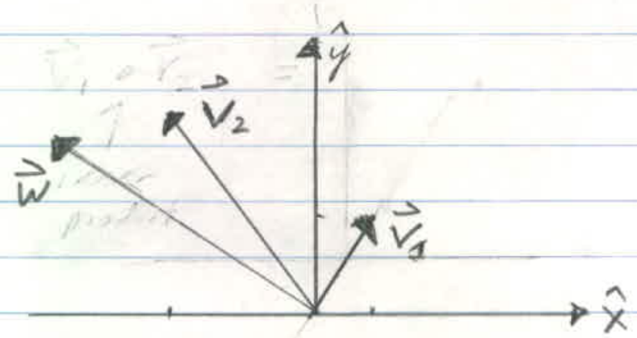
$b = -x+\frac{1}{2}y$

$a = x+3b$

$= -2x+\frac{3}{2}y$

$'' = (-2x+\frac{3}{2}y)\begin{pmatrix}1\\2\end{pmatrix} + (-x+\frac{1}{2}y)\begin{pmatrix}-3\\4\end{pmatrix}$

Spans $\mathbb{R}^2$. But $\vec{V}_2$ is not normal to $\vec{V}_1$. Can we form a vector $\vec{w}$ that is normal to $\vec{V}_1$?



Construct $\vec{W}$ so that it contains only the component of $\vec{V}_2$ that is normal, or perpendicular, to $\vec{V}_1$.

First, a short detour on <u>inner products</u>

$$\vec{V}_1 \cdot \vec{V}_2 = V_1^x V_2^x + V_1^y + V_2^y$$

$$= (1)(-3) + (2)(4) = 5$$

<u>or</u>

$\vec{V}_1 \cdot \vec{V}_2 = \vec{V}_1^T \vec{V}_2$, where "T" means transpose and exchanges row and column entries.

$$" = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

The length or norm of $\vec{V}_1$, denoted $\|\vec{V}_1\|$, is found by

$$\|\vec{V}_1\| \equiv \sqrt{\vec{V}_1 \cdot \vec{V}_1} = \sqrt{(1)(1) + (2)(2)} = \sqrt{5}$$

Similarly

$$\|\vec{V}_2\| = \sqrt{(-3(-3) + (4)(4)} = 5$$

Simple, wait
$\vec{w} \cdot \vec{V}_1 = 0$
let $\vec{w} = V_2 - \alpha V_1$
$\therefore \vec{V_2} \cdot \vec{V_1} - \alpha |V_1|^2 = 0$
$\alpha = \frac{\vec{V_2} \cdot \vec{V_1}}{|\vec{V_1}|^2}$
$\vec{W} = \vec{V}_2 - \frac{\vec{V_2} \cdot \vec{V_1}}{|V_1|^2} \vec{V}_1$

The procedure for finding an orthogonal basis is called the "Gram-Schmidt" process; we consider it in $\mathbb{R}^2$.

$$\vec{w} = \vec{V}_2 - \frac{\vec{V}_2 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1} \vec{V}_1$$

sometimes written $|\vec{V}_1|^2$

fraction of $\vec{V}_1$ that lines along $\vec{V}_2$

$$\vec{w} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} - \frac{5}{(\sqrt{5})^2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$'' = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

To check if $\vec{V}_1$ and $\vec{w}$ are orthogonal, we compute $\vec{V}_1 \cdot \vec{w}$

$$\vec{V}_1 \cdot \vec{w} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = -4 + 4 = 0$$

$\vec{V}_1$ and $\vec{w}$ form an orthogonal basis in $\mathbb{R}^2$. We now seek an orthonormal basis such that

$\hat{V}_1$ is a vector parallel to $\vec{V}_1$ but with a length of 1, i.e.,

$$\hat{V}_1 \equiv \frac{\vec{V}_1}{\|\vec{V}_1\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Similarly

$$\hat{w} = \frac{\vec{w}}{\|w\|} = \frac{1}{\sqrt{20}}\begin{pmatrix}-4\\2\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix}$$

- In general, if $\hat{e}$ is an orthonormal basis in $\mathbb{R}^n$, then any vector in $\mathbb{R}^n$ may be expanded as:

$$\vec{v} = \sum_{i=1}^{n} \underbrace{(\vec{v}\cdot\hat{e}_i)}_{\text{expansion coefficient}} \hat{e}_i$$

Norms are conserved, i.e., Parseval's Theorem

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

$$\text{"} = \sqrt{\left[\sum_{i=1}^{n} (\vec{v}\cdot\hat{e}_i)\hat{e}_i\right]\cdot\left[\sum_{j=1}^{n} (\vec{v}\cdot\hat{e}_j)\hat{e}_j\right]}$$

$$\text{"} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} (\vec{v}\cdot\hat{e}_i)(\vec{v}\cdot e_j)\, \underbrace{\hat{e}_i\cdot\hat{e}_j}_{\delta_{ij}}}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Change of basis preserves power.

$$\text{"} = \sqrt{\underbrace{\sum_{i=1}^{n} (\vec{v}\cdot\hat{e}_i)^2}_{\text{sum of squares of coefficients, or "power" in new basis.}}}$$

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- Matrices are an ordered set of numbers arranged as raws and columns that can be used to transform a vector

Example $$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$ ← raw

↑ column

A is a 2 by 3 (row by column) matrix. It can be used to multiply a vector that has 3 rows to it's right or to multiply a transformed vector, with two columns, to it's left.

Let $\vec{V} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then

$$A\vec{V} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+0-5 \\ 2+0-6 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

2 by 3   3 by 1   2 by 1 vector

must match

1 by 2   2by3

Question: what does this imply for vector arithmetic?

$$v^TA = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$'' = \begin{pmatrix} 1-2 & 3-4 & 5-6 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix}$$

$$(v^TA)^T = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$ is 3 by 1

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- In general, if $\vec{w} = \mathbf{A}\vec{v}$, the elements of $\vec{w}$ are found from

$$w_i = \sum_{k=1}^{n} \underbrace{a_{ik}}_{\text{element of } A} v_k$$

A short detour on transpose

$(A+B)^T = A^T + B^T$

$(cA)^T = cA^T$ where c is a scalar

$(AB)^T = B^T A^T$

For example, in the previous exercise $(v^T A)^T = A^T v$, which is to say that we can always vectors to the right of a matrix, then transpose.

Finally, and important class of matrices have $A = A^T$. These are called symmetric matrices and, clearly, are square matrices (number of columns equals the number of rows)

- Matrices can also multiply matrices.

$$\text{Let } \mathbf{A} = \underset{2\text{ by }3}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}} \text{ and } \mathbf{B} = \underset{3\text{ by }2}{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}}$$

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$$\text{Then } AB = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

a 2by3 multiplied by a 3by2 matrix yields a 2x2 matrix.

$$= \begin{pmatrix} 1-3+0 & 0+3+5 \\ 2-4+0 & 0+4+6 \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ -2 & 10 \end{pmatrix}$$

$$\text{and } BA = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

a 3by2 multiplied by a 2x3 matrix yields a 3by3 matrix

$$= \begin{pmatrix} 1+0 & 3+0 & 5+0 \\ -1+2 & -3+4 & -5+6 \\ 0+2 & 0+4 & 0+6 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

- In general, for $C = AB$, the elements of C are

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where n is the number of colums in A or equivolently the number of rows in B

- We now switch to square matrices, a special but important case. Let's understand how a matrix acts to change a vector.

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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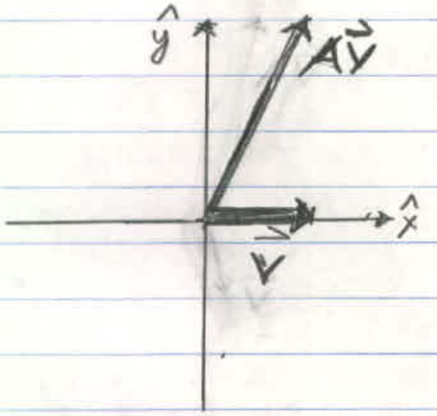

Change in direction and change in length

$$A\vec{v} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Can we find a set of vector such that $A\vec{v}$ changes the length, but not the direction? That is, $A\vec{v} = \underset{\text{scalar}}{\lambda}\,\vec{v}$.

Consider $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $A\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= (-1)\,\vec{v}_1$

Similarly $A\vec{v}_2 = 2\vec{v}_2$

These are the right hand eigenvalues.

They satisfy $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathbb{1})\vec{v} = 0$

eigenvector
eigenvalue
identity matrix

$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let's develop tools to solve this in general.

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Trivial case is $\vec{V}=0$. Otherwise we have

$$\left.\begin{aligned}(a_{11}-\lambda)V_1 + a_{12}V_2 &= 0\\ a_{21}V_1 + (a_{22}-\lambda)V_2 &= 0\end{aligned}\right\} \text{ Solve for } \lambda.$$

$$\vec{V}_2 = -\left(\frac{a_{11}-\lambda}{a_{12}}\right)\vec{V}_1$$

quadratic equation for λ

$$(a_{11}-\lambda)(a_{22}-\lambda)\cancel{\vec{V}_1} - a_{12}a_{21}\cancel{\vec{V}_1} = 0$$

In general, solve n-th order polynomial, noted as the determinate of matrix $\mathbf{A}-\lambda\mathbb{1}$ which, since the rows of $\mathbf{A}-\lambda\mathbb{1}$ are not linearly independent, is set to zero.

This is written as $\det(\mathbf{A}-\lambda\mathbb{1})=0$ or as $|\mathbf{A}-\lambda\mathbb{1}|=0$.

Let's do another example, this time with a real symmetric matrix.

For $\mathbf{A}$ real & symmetric, the eigenvalues are real and at least one eigenvector is real.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Then $\det(\mathbf{A}-\lambda\mathbb{1}) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$

$$'' \quad = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$

Plug back into $\mathbf{A}\vec{V} = \lambda\vec{V}$ to get the corresponding eigenvectors (with a sign)

For $\lambda_1 = 3$, $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

Since rows are linearly dependent, we pick one:

$$2x - y = 3x \quad \text{or} \quad x = -y$$

$$\therefore \ \vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Of course $\underset{\uparrow \text{ constant}}{c}\vec{V}_1$ is also an eigenvector, so

$$\hat{V}_1 = \frac{1}{\|\vec{V}_1\|}\vec{V}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

For $\lambda_2 = 1$ $\quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$

for which $2x - y = x$ or $x = y$

$$\therefore \ \hat{V}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Note: For **A** not symmetric, the eigenvalues are not orthogonal

It is also clear that $\hat{V}_1 \cdot \hat{V}_2 = -\frac{1}{2} + \frac{1}{2} = 0$

The eigenvectors can be used to for a transformation matric, denoted **T**, so that

$$\Lambda \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad .$$

where $\mathbf{T} = \begin{pmatrix} \hat{V}_1 & \hat{V}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

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First, what is $\mathbf{T}^{-1}$? A long digression

What is the inverse of a matrix? Given a matrix $\mathbf{A}$, it may have an inverse, $\mathbf{B}$, where

$$\mathbf{A}\mathbf{B} = \mathbb{1}$$

$\mathbf{B} = \mathbf{A}^{-1}$ if the determinant of $\mathbf{A}$ is nonzero.

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Gauss-Jordan ellimination to get A^{-1}

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \qquad \text{Augment } \mathbf{A} \text{ with } \mathbb{1}$$

$$\downarrow \text{Row 2} \rightarrow \text{Row 2} - 3 \times \text{Row 1}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\downarrow \text{Row 1} \rightarrow \text{Row 1} + \text{Row 2}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\downarrow \text{Row 2} \rightarrow -1/2 \times \text{Row 2}$$

$$\left(\begin{array}{cc|cc} +1 & 0 & -2 & 1 \\ 0 & +1 & 3/2 & -1/2 \end{array}\right)$$

Transformed $\mathbf{A}$ has an inverse as there are there are no zero's on the diagonal, so $\det(A) \neq 0$

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A matrix for which $\det(\mathbf{A})=0$, which is equivalent to having one or more zero on the diagonal after row and column operations, is called signular.

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is clearly singular.

As a check of our example

$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix}$$

$$'' \quad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For the case of a 2 by 2 matrix, a simple way to find $\mathbf{A}^{-1}$ is by Cramers rule, i.e., for

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

There are other ways to look at the determinate.

① Expand $\mathbf{A}$ as $\mathbf{A} = \mathbf{L}\,\mathbf{U}$

square

$$\mathbf{L} = \begin{pmatrix} 1 & & 0 \\ l_{21} & 1 & \\ \vdots & \cdots & \ddots \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & \cdots \\ & u_{22} & \\ 0 & & \ddots \end{pmatrix}$$

where $\mathbf{L}$ has ones on the diagonal and zeros above the diagonal and $\mathbf{U}$ has zeros below the diagonal.

Example $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}$

$\qquad\qquad\mathbf{A} \qquad\qquad \mathbf{L} \qquad\qquad \mathbf{U}$

In general, $\det(\mathbf{A}) = \underset{\substack{\uparrow \\ \text{number of row and column operation} \\ \text{in } \mathbf{L}\text{-}\mathbf{U} \text{ decomposition of } \mathbf{A}}}{(-1)^k} \prod_{i=1}^{n} u_{ii}$

here $k = 2$ and $\det(\mathbf{A}) = (-1)^2 a(d - bc/a)$

$\qquad\qquad\qquad '' = ad - bc$

(2) A geometrical interpretation is that the determinate is the volume of the parallelpiped (parallelogram in $\mathbb{R}^2$) formed by the eigenvectors. For orthogonal eigenvectors, this is a hyper rectangle.

Back to $\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\mathbf{T}^{-1} = \frac{1}{(-1)} \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & -1 \\ -1 & -1 \end{pmatrix} = \mathbf{T}$$

$$\mathbf{T}\mathbf{T}^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbb{1}$$

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$$\Lambda = T^{-1} A T$$

$$\text{"} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\text{"} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\text{"} = \frac{1}{2}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\text{"} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

In general, this is a nice way to transform to a basis set with a single non-zero element.

$$A\vec{V} = \lambda \vec{V} \Rightarrow T^{-1}A\vec{V} = \lambda T^{-1}\vec{V}$$

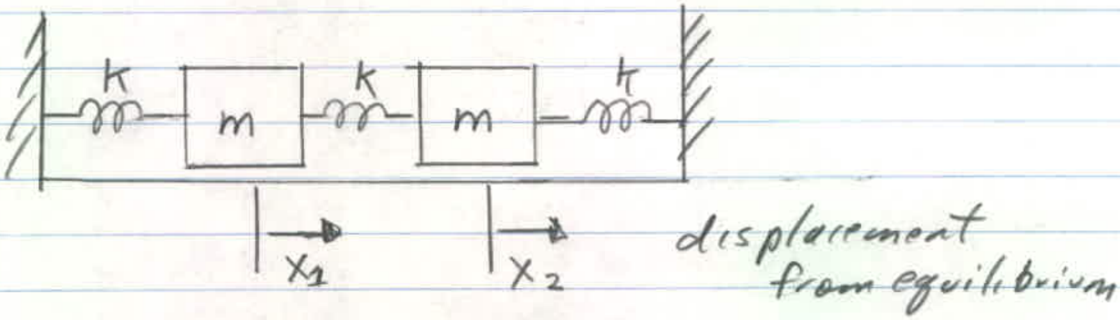
$$\text{but } \Lambda T^{-1} = T^{-1}A$$

$$\therefore \Lambda \underbrace{T^{-1}\vec{V}}_{\text{rotated eigen vector}} = \lambda T^{-1}\vec{V}$$

By design, the columns of T form an orthonormal basis, for which $T^{-1} = T^T$ (special case)

$$\text{so } T^{-1}\hat{V}_1 = T^T\hat{V}_1 = \begin{pmatrix} \hat{V}_1^T \\ \hat{V}_2^T \\ \vdots \end{pmatrix}\hat{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \text{etc.}$$

- Why is the eigenvalue problem with symmetric matrices important? It comes up with physical systems.



$$m\ddot{x}_1 = -k x_1 - k(x_1 - x_2)$$

↑ restoring force

$$m\ddot{x}_2 = -k x_2 - k(x_2 - x_1)$$

Define $\omega_0^2 = k/m$

Take as ansatz $x = x_0 \cos(\omega t + \phi)$

$$\therefore \quad \ddot{x} = -\omega^2 x$$

$$-\omega^2 x_1 = -\omega_0^2 x_1 - \omega_0^2 (x_1 - x_2)$$

$$-\omega^2 x_2 = -\omega_0^2 x_2 - \omega_0^2 (x_2 - x_1)$$

$$\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\text{This is "A"}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↑ eigenvalue ("λ")

$$|A - \omega^2 \mathbb{1}| = \begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

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$$(2\omega_0^2 - \omega^2)^2 - (\omega_0^2)^2 = 0$$

$$(\omega^2)^2 - 4\omega_0^2\omega^2 + 3(\omega_0^2)^2 = 0$$

$$(\omega^2 - 3\omega_0^2)(\omega^2 - \omega_0^2) = 0$$

$$\text{eigenvalues} = 3\omega_0^2,\ \omega_0^2$$

For $\omega^2 = 3\omega_0^2$ $\quad \hat{V}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\omega^2 = \omega_0^2$ $\quad \hat{V}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\therefore\ \mathbf{T}^{-1}\hat{V}_1 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$'' \quad = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{T}^{-1}\hat{V}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$'' \quad = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So $\mathbf{T}$ rotates eigenvectors to lie along cartesian axes. Physically the relevant coordinates are $x_1 + x_2$ (center of mass) and $x_2 - x_1$ (relative motion)