1 A Two Dimensional Neuron

In this problem we will study the dynamics of a neuron described by a membrane potential variable $v$ and a recovery variable $w$. The dynamics of the neuron are in general:

$$\frac{d}{dt} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} f(v, w) \\ g(v, w) \end{pmatrix}.$$  \hspace{1cm} (1)

Assume that for all values of $v$ and $w$:

$$\frac{\partial f(v, w)}{\partial w} < 0 \quad \frac{\partial g(v, w)}{\partial w} < 0 \quad \frac{\partial g(v, w)}{\partial v} > 0$$  \hspace{1cm} (2)

An equilibrium point $(v^*, w^*)$ is the point where $\dot{v} = f(v^*, w^*) = 0$ and $\dot{w} = g(v^*, w^*) = 0$.

1.1 Suppose that at the equilibrium point $\frac{\partial f}{\partial v} < 0$. Show that the equilibrium is stable.

Nullclines are the lines where either $\dot{v}$ or $\dot{w}$ are 0. If the nullclines intersect, that is an equilibrium point. We will call the $f = 0$ line the “$v$ nullcline” (because $\dot{v} = 0$ on that line) and the $g = 0$ line the “$w$ nullcline”.

A specific model

We will work with a Fitzhugh-Nagumo model of the form:

$$f(v, w) = v - \frac{1}{3}v^3 - w + I$$  \hspace{1cm} (3)
$$g(v, w) = \phi(aw - w)$$  \hspace{1cm} (4)

1.2 Plot the nullclines of the model (solve for $w(v)$ such that $f$ or $g$ are 0, see sketch below) for the parameters $I = 3$, $a = 2$. Note that the $v$ nullcline has three “branches” and the $w$ nullcline is monotonic.

1.3 Find parameters such that the equilibrium is in the middle branch of the $v$ nullcline. Compute $\frac{\partial f}{\partial v}$ at the equilibrium and show that it is positive. Given your answer to 1.1, what does that say about the equilibrium point?

1.4 Now find two sets of parameters and plot the nullclines for each of them:

- one such that the equilibrium is in the middle branch of the $v$ nullcline and the slope of the $v$ nullcline is smaller than the slope of the $w$ nullcline at the equilibrium point.
- another such that the equilibrium is in the middle branch of the $v$ nullcline and the slope of the $v$ nullcline is greater than the slope of the $w$ nullcline at the equilibrium point.

Use linear stability analysis to show that in the first case the equilibrium is a node and that in the second case it is a saddle. Can you use the nullcline plots to explain this graphically?
1.5 Run the dynamics. Plot example spike trains and discuss the different regimes you identified above.

1.6 (Bonus) A general model. Assume the $v$ nullcline has a cubic shape and the $w$ nullcline is monotonically increasing (see sketch). Prove that if there is an equilibrium point in the middle branch of the $v$ nullcline then $\frac{\partial f}{\partial v} > 0$.

2 Noise and “Balanced” Network

Assume that a neuron in a neural network receives both $K$ excitatory and $K$ inhibitory inputs from the presynaptic neurons, each of which spikes ($V_j = 1$) with probability $m$ and silences ($V_j = 0$) with probability $1 - m$. The total input sending to the postsynaptic neuron is

$$\mu = \sum_{j=1}^{K} W_j^E V_j^E + \sum_{i=1}^{K} W_i^I V_i^I,$$

where the weights, $W_j^E(K)$ and $W_i^I(K)$, depend on the number of connections $K$. We would like to explore the relation between the mean $\langle \mu \rangle$, the variance of total input $\text{Var}(\mu)$ and the number of presynaptic connections ($K$) with numerical simulations, given two scenarios:

(1) A postsynaptic neuron receives purely excitatory inputs from $K$ preynaptic neurons ($W_j^E \neq 0, W_j^I = 0$);

(2) A postsynaptic neuron receives both $K$ excitatory and $K$ inhibitory inputs ($W_j^{E,I} \neq 0$).

2.1 Consider the case of purely $K$ excitatory inputs ($W_j^E = \frac{1}{K} \neq 0, W_j^I = 0$),

$$\mu = \sum_{j=1}^{K} W_j^E V_j^E = \frac{1}{K} \sum_{j=1}^{K} V_j^E.$$

To calculate the mean $\langle \mu \rangle$ and the variance $\text{Var}(\mu)$ of the total input $\mu$, firstly we need to generate $K$ excitatory inputs for one time step, and extend it to a sequence (i.e., multiple time steps) over the time duration $T$. Then, we can calculate the average $\langle \mu \rangle$ and the variance $\text{Var}(\mu)$ from this input sequence. Here are the steps for you to follow:

1 Problem courtesy of Bard Ermentrout. A good resource that can help with some background is chapter 3 of Bard’s book: Mathematical Foundations of Neuroscience. The eBook is available for free through http://roger.ucsd.edu/
STEP 1: Generate $K$ excitatory inputs occurring in one time step ($dt$) as a one-dimensional binary vector

$$V^E = (V^E_1, V^E_2, \cdots) = (1, 0, 0, 0, 1, \cdots)^T.$$ 

*Hint*: use MATLAB command $(\text{rand}(K, 1) <= m \times dt)$ to generate a $K$-by-1 column vector, where $m \times dt$ is the spiking probability of one time bin ($dt$). You can choose the spiking probability ($m$) between $0 < m \leq 1$.

STEP 2: Extend the $K$ excitatory inputs ($K$-by-1) from one-time step to a sequence over a period of time $T \gg dt$ (ex: you may set $dt = 1$ and $T = 10000$). *Hint*: how many $dt$ bins ($n$) are there in the time interval $T$? Find the number of bins ($n$) and use $(\text{rand}(K, n) <= m \times dt)$ to generate a binary $K$-by-$n$ matrix. Each column represents $K$ inputs at each time step, while each row is the input sequence sending from one of the presynaptic neurons.

STEP 3: From the $K$-by-$n$ matrix, how would you get a total input $\mu$ received at each time step? *Hint*: $(1/K) \times \text{sum}(\text{rand}(K, n) <= m \times dt, 1)$, and it will be a 1-by-$n$ row vector.

STEP 4: Now you have input sequence $\mu$ (1-by-$n$ row vector), you will be able to calculate the mean $\langle \mu \rangle$ and the variance $\text{Var}(\mu)$. *Hint*: You may need some built-in MATLAB functions. Try to look them up.

STEP 5: Write a for-loop that calculate the mean $\langle \mu \rangle$ and variance $\text{Var}(\mu)$ with different connections $K = 200, 400, 600, 800, 1000, 1500, 2000, 3500, \text{and} 5000$. For each $K$, you may want to run multiple trials and take the average.

STEP 6: Plot $\langle \mu \rangle$ vs $K$, and $\text{Var}(\mu)$ vs $K$. Compare each curve with its own zero (horizontal line $y = 0$, label it on the plot would be helpful), and briefly describe the trend for each.

2.2 $K$ excitatory and $K$ inhibitory inputs ($W^E_j = -W^I_j = \frac{1}{\pi}$),

$$\mu = \frac{1}{K} \left( \sum_{j=1}^{K} V^E_j - \sum_{i=1}^{K} V^I_i \right).$$

Extend the simulation steps in 2.1, plot $\langle \mu \rangle$ vs $K$ and $\text{Var}(\mu)$ vs $K$ for a case that a neuron receives both $K$ excitatory and $K$ inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), and briefly describe the trend for each.

2.3 $K$ excitatory and $K$ inhibitory inputs, with the weights $W^E_j = -W^I_j = \frac{1}{\sqrt{K}}$,

$$\mu = \frac{1}{\sqrt{K}} \left( \sum_{j=1}^{K} V^E_j - \sum_{i=1}^{K} V^I_i \right).$$

Extend the simulation steps in 2.1, plot $\langle \mu \rangle$ vs $K$ and $\text{Var}(\mu)$ vs $K$ for a case that a neuron receives both $K$ excitatory and $K$ inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), briefly describe the trend for each, and compare them with the results you got from 2.1 and 2.2.