Winter 2017 PHYS 178/278, Homework 2
Due 9:30 AM on February 7^{th}

Please turn in a hard copy of your homework to TA by the end of the lecture!

1 A Two Dimensional Neuron[1]

In this problem we will study the dynamics of a neuron described by a membrane potential variable v and a recovery variable w. The dynamics of the neuron are in general:

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} f(v,w) \\ g(v,w) \end{pmatrix}. \tag{1}$$

Assume that for all values of v and w:

$$\frac{\partial f(v,w)}{\partial w} < 0 \qquad \frac{\partial g(v,w)}{\partial w} < 0 \qquad \frac{\partial g(v,w)}{\partial v} > 0 \tag{2}$$

An equilibrium point $(v^\star, w^\star)$ is the point where $\dot{v} = f(v^\star, w^\star) = 0$ and $\dot{w} = g(v^\star, w^\star) = 0$.

1.1 Suppose that at the equilibrium point $\frac{\partial f}{\partial v} < 0$. Show that the equilibrium is stable.

Nullclines are the lines where either $\dot{v}$ or $\dot{w}$ are 0. If the nullclines intersect, that is an equilibrium point. We will call the $f = 0$ line the "v nullcline" (because $\dot{v} = 0$ on that line) and the $g = 0$ line the "w nullcline".

A specific model

We will work with a Fitzhugh-Nagumo model of the form:

$$f(v,w) = v - \frac{1}{3}v^3 - w + I \tag{3}$$

$$g(v,w) = \phi(av - w) \tag{4}$$

1.2 Plot the nullclines of the model (solve for $w(v)$ such that f or g are 0, see sketch below) for the parameters $I = 3$, $a = 2$. Note that the v nullcline has three "branches" and the w nullcline is monotonic.

1.3 Find parameters such that the equilibrium is in the middle branch of the v nullcline. Compute $\frac{\partial f}{\partial v}$ at the equlibrium and show that it is positive. Given your answer to **1.1,** what does that say about the equilibrium point?

1.4 Now find two sets of parameters and plot the nullclines for each of them:

- one such that the equilibrium is in the middle branch of the v nullcline **and** the slope of the v nullcline is **smaller** than the slope of the w nullcline at the equilibrium point.
- another such that the equilibrium is in the middle branch of the v nullcline **and** the slope of the v nullcline is **greater** than the slope of the w nullcline at the equilibrium point.

Use linear stability analysis to show that in the first case the equilibrium is a node and that in the second case it is a saddle. Can you use the nullcline plots to explain this graphically?

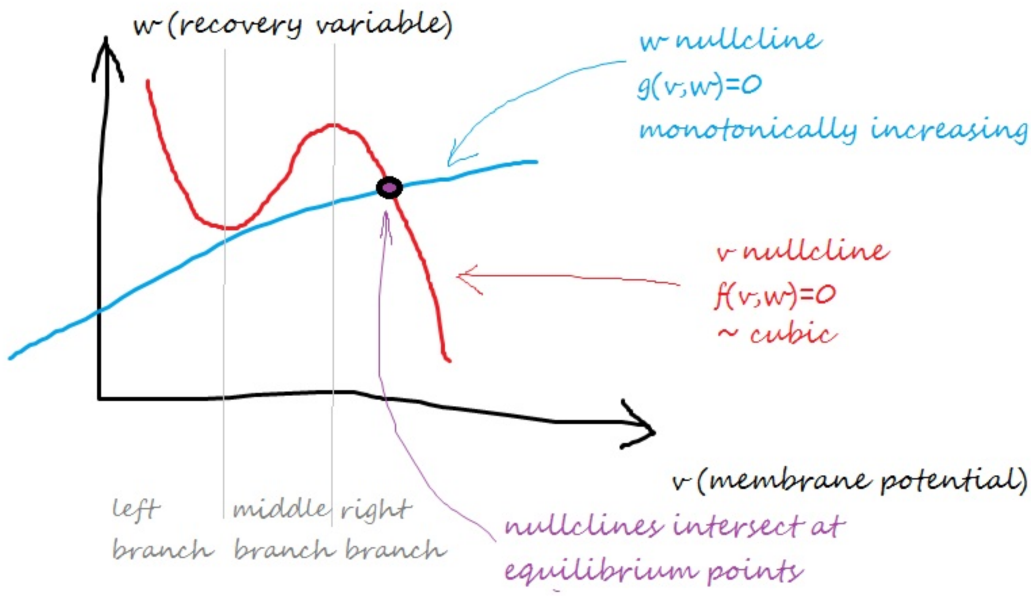

Figure 1: Sketch of v, w nullclines.

1.5 Run the dynamics. Plot example spike trains and discuss the different regimes you identified above.

1.6 (Bonus) A general model. Assume the v nullcline has a cubic shape and the w nullcline is monotonically increasing (see sketch). Prove that if there is an equilibrium point in the middle branch of the v nullcline then $\frac{\partial f}{\partial v} > 0$.

2 Noise and "Balanced" Network

Assume that a neuron in a neural network receives both K excitatory and K inhibitory inputs from the presynaptic neurons, each of which spikes ($V_j = 1$) with probability m and silences ($V_j = 0$) with probability $1 - m$. The total input sending to the postsynaptic neuron is

$$\mu = \sum_{j=1}^{K} W_j^E V_j^E + \sum_{i=1}^{K} W_i^I V_i^I,$$

where the weights, $W_j^E(K)$ and $W_j^I(K)$, depend on the number of connections K. We would like to explore the relation between **the mean** $\langle\mu\rangle$, **the variance of total input Var**(μ) and **the number of presynaptic connections** (K) with numerical simulations, given two senarios:
(1) A postsynaptic neuron receives purely excitatory inputs from K preynaptic neurons ($W_j^E \neq 0$, $W_j^I = 0$);
(2) A postsynaptic neuron receives both K excitatory and K inhibitory inputs ($W_j^{E,I} \neq 0$).

2.1 Consider the case of purely K excitatory inputs ($W_j^E = \frac{1}{K} \neq 0$, $W_j^I = 0$),

$$\mu = \sum_{j=1}^{K} W_j^E V_j^E = \frac{1}{K} \sum_{j=1}^{K} V_j^E.$$

To calculate the mean $\langle\mu\rangle$ and the variance Var(μ) of the total input μ, firstly we need to generate K excitatory inputs for one time step, and extend it to a sequence (i.e., multiple time steps) over the time duration T. Then, we can calculate the average $\langle\mu\rangle$ and the variance Var(μ) from this input sequence. Here are the steps for you to follow:

[1] Problem courtesy of Bard Ermentrout. A good resource that can help with some background is chapter 3 of Bard's book: Mathematical Foundations of Neuroscience. The eBook is available for free through http://roger.ucsd.edu/

STEP 1: Generate K excitatory inputs occurring in one time step (dt) as a one-dimensional binary vector

$$V^E = \left(V_1^E, V_2^E, \cdots\right) = (1, 0, 0, 0, 1, \cdots)\,.^T$$

Hint: use MATLAB command ($\texttt{rand(K,1)} <= \texttt{m} * \texttt{dt}$) to generate a K-by-1 column vector, where $\texttt{m} * \texttt{dt}$ is the spiking probability of one time bin (dt). You can choose the spiking probability (m) between $0 < m \leq 1$.

STEP 2: Extend the K excitatory inputs (K-by-1) from one-time step to a sequence over a period of time $T \gg dt$ (ex: you may set $dt = 1$ and $T = 10000$). *Hint*: how many dt bins (n) are there in the time interval T? Find **the number of bins (n)** and use ($\texttt{rand(K,n)} <= \texttt{m} * \texttt{dt}$) to generate a binary K-by-n matrix. Each column represents K inputs at each time step, while each row is the input sequence sending from one of the presynaptic neurons.

STEP 3: From the K-by-n matrix, how would you get a total input μ received at each time step?
Hint: $\texttt{(1/K)} * \texttt{sum(rand(K,n)} <= \texttt{m} * \texttt{dt, 1)}$, and it will be a 1-by-n row vector.

STEP 4: Now you have input sequence μ (1-by-n row vector), you will be able to calculate **the mean** $\langle\mu\rangle$ and **the variance Var** (μ). *Hint*: You may need some built-in MATLAB functions. Try to look them up.

STEP 5: Write a for-loop that calculate the mean $\langle\mu\rangle$ and variance $\mathrm{Var}(\mu)$ with different connections $K = 200$, 400, 600, 800, 1000, 1500, 2000, 3500, and 5000. For each K, you may want to run multiple trials and take the average.

STEP 6: Plot **$\langle\mu\rangle$ vs K**, and **$\mathrm{Var}(\mu)$ vs K**. Compare each curve with its own zero (horizontal line $y = 0$, label it on the plot would be helpful), and briefly describe the trend for each.

2.2 K excitatory and K inhibitory inputs ($W_j^E = -W_j^I = \frac{1}{K}$),

$$\mu = \frac{1}{K}\left(\sum_{j=1}^{K} V_j^E - \sum_{i=1}^{K} V_i^I\right).$$

Extend the simulation steps in **2.1**, plot $\langle\mu\rangle$ vs K and $\mathrm{Var}(\mu)$ vs K for a case that a neuron receives both K excitatory and K inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), and briefly describe the trend for each.

2.3 K excitatory and K inhibitory inputs, with the weights $W_j^E = -W_j^I = \frac{1}{\sqrt{K}}$,

$$\mu = \frac{1}{\sqrt{K}}\left(\sum_{j=1}^{K} V_j^E - \sum_{i=1}^{K} V_i^I\right).$$

Extend the simulation steps in **2.1**, plot $\langle\mu\rangle$ vs K and $\mathrm{Var}(\mu)$ vs K for a case that a neuron receives both K excitatory and K inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), briefly describe the trend for each, and compare them with the results you got from **2.1** and **2.2**.