Phys 178: HW2

Ghita Guessous

February 2024

Due midnight on March. 4th. Please justify all of your answers, make intermediate plots and submit to Gradescope. Make sure you select the correct question for each part of your submission, otherwise it will be graded as missing. If you have any questions, please email Ghita Guessous (gguessou@ucsd.edu).

- All physics students (graduate and undergrad) are required to do both Problems.
- Undergraduate biology students are required to do Problems 1.1 and 2 but are encouraged to attempt the rest

1 Poisson Distribution

An integer-valued random variable X has a Poisson distribution with parameter λ (i.e: X $P(\lambda, k)$) if the probability that $X = k$ is:

$$\mathbb{P}\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda} \quad \text{for} \quad k = 0, 1, 2, ...$$

The real number $\lambda > 0$ is called the rate parameter of the distribution.

1.1 Suppose X_1 and X_2 are independent random variables and have Poisson distributions with parameters λ_1 and λ_2. **Show** that the random variable $X = X_1 + X_2$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

1.2 Consider a sequence of independent Bernoulli random variables $\{Y_n\}_{n=1}^N$, with

$$\mathbb{P}\{Y_n = k\} = \begin{cases} p_n & \text{if} \quad k = 1, \\ 1 - p_n & \text{if} \quad k = 0, \end{cases} \quad \forall n.$$

Each Y_n takes values from 0 and 1 and can be imagined as the occurrence of an event. Assume that there are many possible events, $N \to +\infty$, but each event is rare,

$$\lim_{N\to+\infty} \max_{1\leq n\leq N} p_n \to 0,$$

such that the overall probability of the occurrences is finite,

$$\lim_{N\to+\infty} \sum_{n=1}^{N} p_n \to \lambda \in (0, +\infty).$$

Show that the total number of events, $S_N = \sum_{n=1}^N Y_n$, satisfies a Poisson distribution with parameter λ as $N \to +\infty$.

[Hint: One way to do this is to use a moment generating function. Here is an outline:

The generating function for Y_n is $f_n(x) = 1 + (x-1)p_n$. So the generating function for S_N is

$$F_N(x) = \prod_{n=1}^{N} f_n(x).$$

Use $\ln[1 + (x-1)p_n] = 1 + (x-1)p_n + O(p_n^2)$ to show $F_N(x) \to e^{\lambda(x-1)}$, which is the generating function for the Poisson distribution.]

2 Bursting oscillations [1]

Many neurons exhibit much more complicated firing patterns than simple repetitive firing like we saw in class in the Hodgkin-Huxley and Fitzhugh-Nagumo models. A common mode of firing in many neurons are bursting oscillations (Figure 1). The following Hindmarsh-Rose model can describe neurons with bursting firing pattern:

$$\begin{aligned}
\frac{dx}{dt} &= y - x^3 + bx^2 - z + I_{app}, \\
\frac{dy}{dt} &= c - dx^2 - y, \\
\frac{dz}{dt} &= \epsilon \left[A(x - x_0) - z \right].
\end{aligned} \tag{1}$$

x is a voltage-like variable. y and z are variables related to ion channels. z is a slow variable and $\epsilon \ll 1$. $b, c, d > 0$ are constants. **Find** a set of parameter values $(b, c, d, A, x_0, I_{app})$ such that Eq.(1) can generate bursting oscillations and **explain** why this set of parameters work.

Below is a way to find these parameters (you don't need to follow this):

2.1 Bifurcation diagram for x vs. z

Since z is a slow variable ($\frac{dz}{dt} \approx 0$), we can treat z as a bifurcation parameter and take $I_{app} = 0$ for convenience. We first investigate the behavior of the 2D system (x, y) for different values of z:

$$\begin{aligned}
\frac{dx}{dt} &= y - x^3 + bx^2 - z \\
\frac{dy}{dt} &= c - dx^2 - y.
\end{aligned} \tag{2}$$

a. Show that when $b^2 < 3$, for any value of z, the 2D system (x, y) doesn't have periodic solutions.

[Hint: use Bendixson criterion.]

Without periodic solutions, the 2D system (Eq. (2)) will always go to a fixed point as $t \to +\infty$. Thus, the original system (Eq. (1)) can't have bursting oscillations. Below we will assume $b^2 > 3$. We further assume $b < d$. [Think about what happens if $b > d$ but you don't need to write down anything.]

b. Show that when z is large enough ($z \to +\infty$), Eq. (2) has only one stable fixed point. Convince yourself this fixed point is a global attractor in this regime. **Find** the maximum value of z such that Eq. (2) has another stable fixed point. Denote this value as $z_{sn,1}$.

[1] This problem is adapted from Hindmarsh and Rose (1984), Proc. R. Soc. Lond. B 221, 87-102.

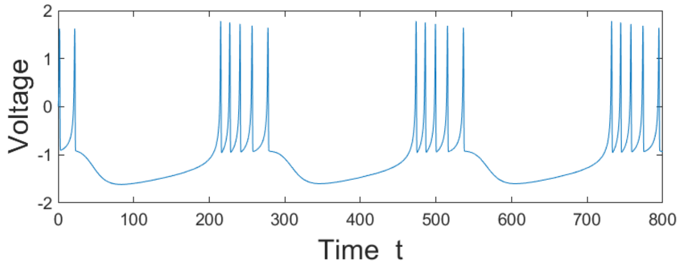

Figure 1: Bursting oscillations

c. When decreasing z further, the newly generated stable fixed point will eventually become unstable. **Find** the maximum value $z_{Hopf,1}$ when this happens. **Show** a stable limit cycle L_1 is generated when $z = z_{Hopf,1}$.

[Hints: Calculate the Lyapunov exponent at the Hopf bifurcation.]

It is difficult to directly analyze what will happen for the stable limit cycle L_1 when further decreasing z. We will investigate this numerically. Before running into numerical simulations, we can also identify part of the bifurcation diagram from the $-z$ direction.

c. **Show** that when z is small enough ($z \to -\infty$), Eq. (2) has only one stable fixed point. Convince yourself this fixed point is a global attractor in this regime. **Find** the minimum value of z such that this fixed point becomes unstable. Denote this value as $z_{Hopf,2}$.

d. **Show** that a stable limit cycle L_2 is generated when $z = z_{Hopf,2}$.

e. When further increasing z, at $z = z_{sn,2} > z_{Hopf,2}$, a new stable fixed point is generated. **Find** the value of $z_{sn,2}$.

[Hints: Calculate the Lyapunov exponent at the Hopf bifurcation.]

Now we are ready to fill in the missing part of the bifurcation diagram through numerically simulation.

f. Let $b = 3$ and $d = 5$. Simulate Eq.(2) for different values of z to complete the bifurcation diagram. **In words**, what happens to the stable limit cycles L_1 and L_2 at the region $z_{sn,2} < z < z_{Hopf,1}$?

[Feel free to try other values of (b, d) for $b < d$.]

2.2 Parameters for bursting oscillations

Based on the bifurcation diagram, **find** a set of parameters $(b, c, d, A, x_0, I_{app})$ such that Eq.(1) can generate bursting oscillations. **Plot** the trajectory $(x(t), y(t), z(t))$ in 3D space and $x(t)$ vs. t.