

Neural variability and Poisson statistics

January 15, 2014

1 Introduction

We are in the process of deriving the Hodgkin-Huxley model. That model describes how an action potential is generated by ion specific and voltage dependent channels in a single neuron. It is a biophysical deterministic model.

Biophysical, because we started from a description of the different voltage gated ion channels and how they act together to create a bistable system.

Deterministic, because the Hodgkin-Huxley equations that you'll see next week have neither inherent variability nor chaotic behavior. If you start them with the same initial conditions and drive the model neuron with the same input, you will always get the same spike train.

When experimentalists make measurements of spike trains in all areas of the nervous system, even if they control the environment very carefully, the neural response to repeated stimuli is variable. This means that the noise in the neural response is real. After the famous Hodgkin-Huxley papers in the 50's, it took people a few years, until the early to mid 60's, to realize that the variability they were seeing may not go hand in hand with the deterministic nature of the Hodgkin-Huxley model.

2 Reconciling Hodgkin-Huxley with neural response variability

One easy way out is simply saying "The neuron is deterministic. The variability in the neuronal response comes from stochasticity in the synaptic inputs". A longer way out would be to say that we should look for a stochastic mechanism that has to be somehow integrated into the biophysical model and generate noise inside the neuron itself.

It's quite rare that the easy way out is the justified one. William Calvin and Chuck Stevens showed in 1967 that it's possible to explain the variability of inter spike intervals with the noise of the synaptic inputs.

This is not to say that all of neural variability is a result of the noise in many synaptic inputs. Effects of recurrent connectivity can sometimes dominate the variability in the response of single neurons (for example in the balanced state van Vreeswijk and Sompolinsky [1996]), or, sometimes lead to the opposite effect of locking to a particular rhythm (many works by Eve Marder and Nancy Kopell, for example Marder and Bucher [2001]). In this lecture, we will focus on the noisiness of a neuron that is in isolation (or part of a purely feedforward circuit), meaning it does not affect its inputs through recurrent connections.

3 Poisson statistics

Once it is established that neuronal firing variability is dominated by the stochasticity of synaptic inputs, the most natural assumption to make is that inputs are independent. In the homework you are asked to discuss why, if there is nonlinear dendritic integration this assumption is not true. Let's also make the assumption that the inputs have a constant average. Since the inputs are constant on average, the output, or the average firing rate is also constant,

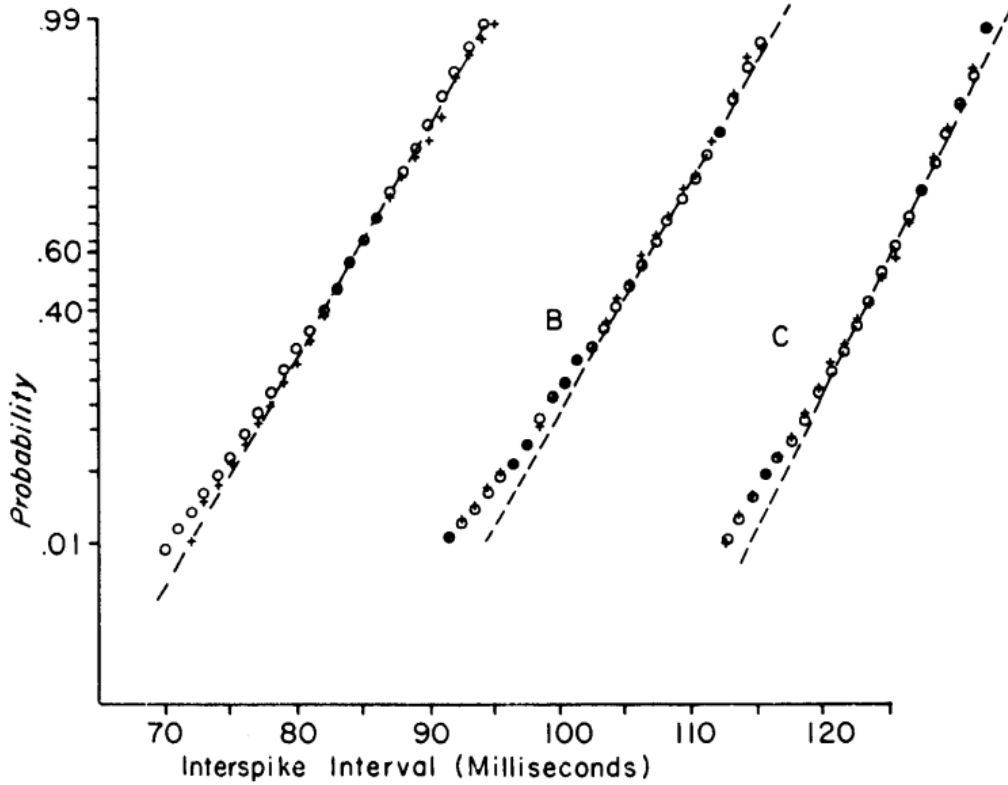


Figure 1: Taken from Calvin and Stevens [1967]

so we will call it r . After we're finished and we have a result for r we will allow for the firing rate to be time dependent $r(t)$.

We want to find the probability that the neuron spiked n times during the period T :

$$P_T(n) \tag{1}$$

We divide the interval T into M bins such that

$$\Delta t = \frac{T}{M}. \tag{2}$$

M is large enough (Δt is small enough) so that there are never two spikes in the same bin.

The noisy inputs allow us to make the following simplifying assumption: having a spike in the i th bin $t_i > t > t_i + \Delta t$ is independent of having or not having a spike in the $i - 1$ bin (or the $i + 1$ bin). Again, in the homework you're asked to discuss specific cases where this assumption does not hold.

Our strategy to find $P_T(n)$ is the following (taken from Dayan and Abbott [2001]). It's a product of three factors:

$$P_T(n) = P \left(\begin{array}{c} n \text{ spikes in} \\ n \text{ specific} \\ \text{bins} \end{array} \right) \times P \left(\begin{array}{c} 0 \text{ spikes in} \\ \text{other } M - n \\ \text{bins} \end{array} \right) \times \left(\begin{array}{c} \# \text{ of ways of} \\ \text{putting } n \text{ spikes} \\ \text{in } M \text{ bins} \end{array} \right) \tag{3}$$

After making all the simplifying assumptions, this is rather easy:

$$P \left(\begin{array}{c} n \text{ spikes in} \\ n \text{ specific} \\ \text{bins} \end{array} \right) = \left(\frac{rT}{M} \right)^n = (r\Delta t)^n \tag{4}$$

$$P \left(\begin{array}{c} 0 \text{ spikes in} \\ \text{other } M-n \\ \text{bins} \end{array} \right) = (1 - r\Delta t)^{M-n} \quad (5)$$

$$\left(\begin{array}{c} \# \text{ of ways of} \\ \text{putting } n \text{ spikes} \\ \text{in } M \text{ bins} \end{array} \right) = \binom{M}{n} = \frac{M!}{(M-n)!n!} \quad (6)$$

Now we have a tiny bit of calculus before we're done. We need to take the limit $\Delta t \rightarrow 0$, or $M \rightarrow \infty$ and $M-n \approx M$. The three factors above become:

1. $(r\Delta t)^n$: nothing to do about this one
2. $\lim_{\Delta t \rightarrow 0} (1 - r\Delta t)^{M-n}$:
 n is constant while $M = \frac{T}{\Delta t} \rightarrow \infty$, so we can approximate $M-n \approx \frac{T}{\Delta t}$:

$$\lim_{\Delta t \rightarrow 0} (1 - r\Delta t)^{M-n} \approx \lim_{\Delta t \rightarrow 0} (1 - r\Delta t)^{\frac{T}{\Delta t}} \quad (7)$$

Next we multiply and divide by T inside the parenthesis:

$$\lim_{\Delta t \rightarrow 0} (1 - r\Delta t)^{\frac{T}{\Delta t}} = \lim_{\Delta t \rightarrow 0} \left(1 - \frac{\Delta t}{T} (rT) \right)^{\frac{T}{\Delta t}} \quad (8)$$

Then we remind ourselves of the definition of the exponential function $\lim_{\epsilon \rightarrow 0} (1 + a\epsilon)^{\frac{1}{\epsilon}} = e^a$, so :

$$\lim_{\Delta t \rightarrow 0} (1 - r\Delta t)^{M-n} = e^{-rT} \quad (9)$$

3. For the last term we need the Stirling approximation, and remembering that it's ok to replace $M-n$ with M (unless it's in the exponent):

$$\binom{M}{n} = \frac{M!}{(M-n)!n!} \quad (10)$$

$$\approx \frac{M^M}{(M-n)^{M-n} n!} \quad (11)$$

$$\approx \frac{M^n}{n!} \quad (12)$$

$$= \left(\frac{T}{\Delta t} \right)^n \frac{1}{n!} \quad (13)$$

combining everything

$$\begin{aligned} P_T(n) &= \frac{(r\Delta t)^n}{n!} \times e^{-rT} \times \left(\frac{T}{\Delta t} \right)^n \\ &= \frac{(rT)^n}{n!} e^{-rT} \end{aligned} \quad (14)$$

which is exactly the Poisson distribution.

If the changes in firing rate are slow compared to the period T :

$$T \frac{d}{dt} r(t) \ll r(T) \quad (15)$$

then we can replace $r \rightarrow r(t)$. Note that we can't make T too small because the number of spikes in a small window T will make the RHS of the inequality small. Dayan and Abbott do this much more carefully, so if among all the assumptions we made this is the one that bothers you, all these calculations are done quite clearly in their book.

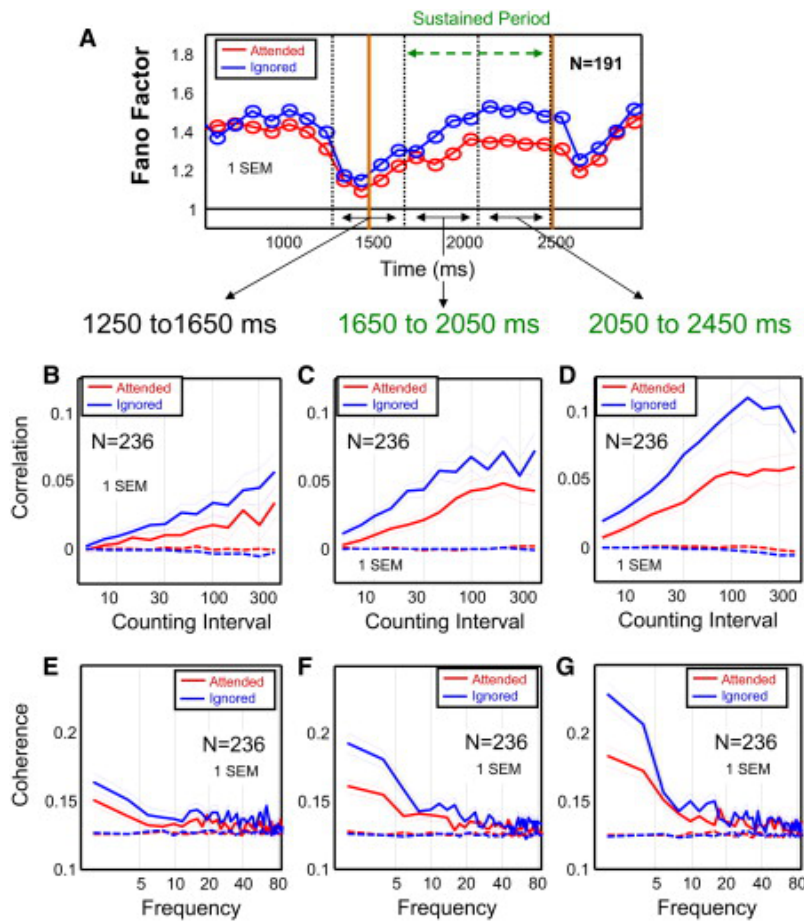


Figure 2: Taken from Mitchell et al. [2009]

4 What is it good for?

A long standing question that many neuroscientists owe their career to is the question of rate code versus temporal code. A rate code means that the information that neurons pass between them is not in the exact timing of single spikes but in the quantity of spikes that arrive within some time window. A temporal code is the opposite. It means that the timing of single spikes is what carries the information. In the homework you are asked to discuss what Poisson firing statistics means in the context of this debate.

One very important thing about the Poisson distribution is that the variance is equal to the mean. Showing this is also a part of your homework. This means that if under some conditions, when we're presenting repeated trials the variance of the spike counts is roughly equal to the mean, we can say with some confidence that the spiking statistics are Poisson. If under other conditions this is no longer true, and the variance is bigger or smaller than the mean, the spiking is no longer Poisson, and we have to go back to see which of the assumptions that we made in our derivation no longer holds and why.

4.1 Attention

A nice example of such modulation was done next door at the lab of John Reynolds at the Salk institute. They found that attention reduces the fano factor - the ratio of the variance and the mean - of neurons in V4 of Macaque monkeys. I won't get into all the details, but just point out that this is strong evidence that the network is working hard to reduce the noise in the network in a dramatic way: the modulation is of the whole noise distribution, not only the noise "amount".

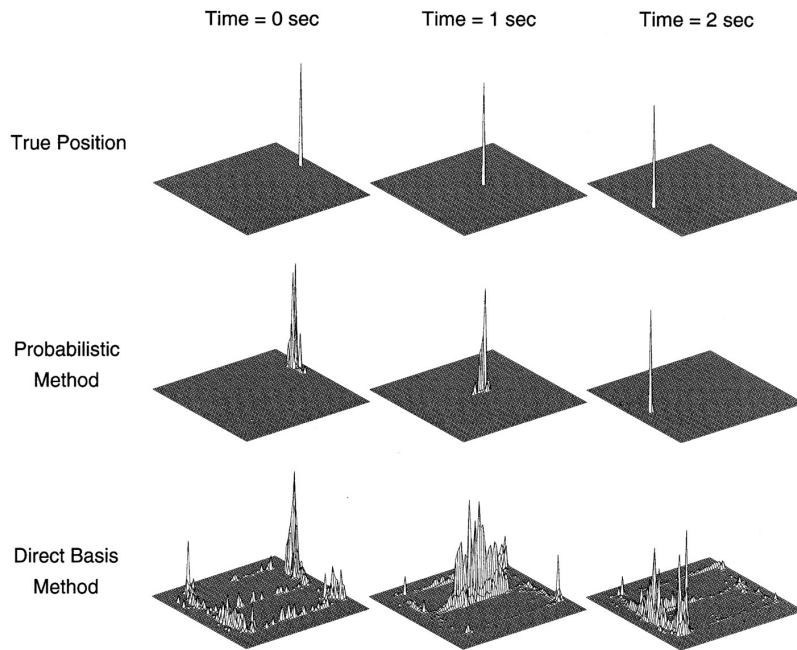


Figure 3: Taken from Zhang et al. [1998]

4.2 Exponential distribution

From a more theoretical point of view, the functional form of the Poisson distribution makes life easier. This is because many times we are interested in quantities like $\frac{d}{d\theta} P(n)$ (for example to compute the Fisher information of a population with some tuning curves for the stimulus variable θ , see Seung and Sompolinsky [1993]) or the derivatives of the log probability (to compute the maximum likelihood solution or to fit a Generalized Linear Model, see Pillow et al. [2005]). For these reasons theorists will many times make the assumption that spiking has Poisson statistics even if it's not 100% true.

5 What are the limitations?

By now it should be quite clear that any number of the assumptions that we made can be unjustified in some biological network.

5.1 Refractory period/ bursts/ dendritic nonlinearities

In the homework you are asked to discuss a refractory period, a tendency to burst firing and/or nonlinear dendritic computations (e.g. coincidence detection) are hard to reconcile with Poisson statistics. (I specifically say “hard” and not “impossible” because if one allows the firing rate to be time dependent there are ways to work around this, for example by fitting a GLM.)

5.2 Independent inputs \nRightarrow Poisson statistics

Preparing for this lecture I came across a relatively recent PRE paper (Lindner [2006]) that claims independent inputs don't necessarily result in Poisson statistics. If you're interested this could be a good starting point for a project.

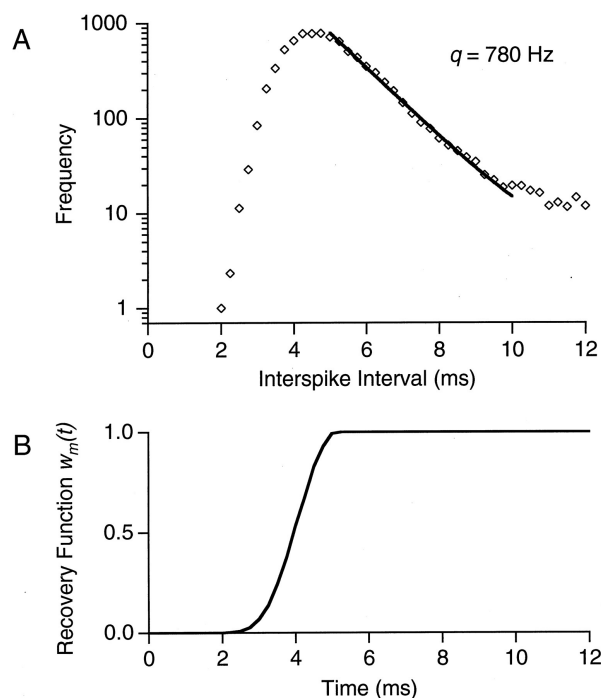


Figure 4: Taken from Berry and Meister [1998]

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