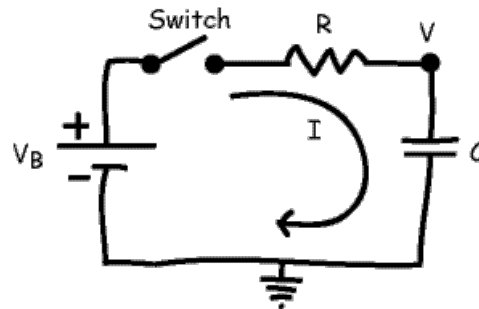


This handout covers current and voltage changes in a single-loop circuit with a resistor and a capacitor. It provides a good model for a cell membrane, whose wall serves as both resistor and capacitor.



For the circuit above the initial charge of the capacitor is zero and the switch is closed at time $t = 0^+$. We have:

$$0 = -V_B + IR + \frac{Q}{C} .$$

Recall that the current is just the change in charge with response to time, i.e., $I = \Delta Q / \Delta t$. This charge must end up on the capacitor plates. So:

$$0 = -V_B + \frac{\Delta Q}{\Delta t} R + \frac{Q}{C} .$$

A little rearranging to place each term in the form of a current gives:

$$\frac{V_B}{R} = \frac{\Delta Q}{\Delta t} + \frac{Q}{\tau} .$$

where $\tau \equiv RC$. To solve this equation, we take the limit that $\Delta t \rightarrow 0$ and write:

$$\frac{V_B}{R} = \frac{dQ}{dt} + \frac{Q}{\tau} .$$

The solution of this equation - recall that $\int_a^b \frac{dx}{x} = \log_e(b) - \log_e(a)$ and - for fun - try to solve it yourself - is:

$$Q(t) = CV_B (1 - e^{-t/\tau}) .$$

Thus $I = dQ/dt$ is:

$$I(t) = \frac{V_B}{R} e^{-t/\tau}$$

and $I(t = 0^+) = V_B/R$ and $I(t \rightarrow \infty) = 0$. With $V = Q/C$, the voltage across the capacitor is:

$$V(t) = V_B (1 - e^{-t/\tau}) .$$

and $V(t = 0^+) = 0$ and $V(t \rightarrow \infty) = V_B$.

A final point concerns the energy stored in the capacitor. The instantaneous power is just:

$$\begin{aligned} P(t) &= I(t) V(t) \\ &= \frac{V_B^2}{R} e^{-t/\tau} (1 - e^{-t/\tau}) \\ &= \frac{V_B^2}{R} (e^{-t/\tau} - e^{-2t/\tau}) \end{aligned}$$

The energy stored after the capacitor charges is the integral of the power, i.e.,

$$E = \int_0^{\infty} dt P(t)$$

This becomes - recall that $\int_a^b dx e^{\lambda x} = \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}$ and - for fun - try to do it yourself! -

$$\begin{aligned} E &= \frac{V_B^2}{R} \int_0^{\infty} dt (e^{-t/\tau} - e^{-2t/\tau}) \\ &= \frac{V_B^2}{R} \left(\frac{e^{-\infty} - e^0}{-1/\tau} - \frac{e^{-\infty} - e^0}{-2/\tau} \right) \\ &= \frac{V_B^2}{R} \left(\frac{0-1}{-1/RC} - \frac{0-1}{-2/RC} \right) \\ &= \frac{V_B^2}{R} \left(RC - \frac{RC}{2} \right) \\ &= \frac{1}{2} C V_B^2 \end{aligned}$$

which is the expected result for energy in a capacitor.

Bottom Line: A resistor/capacitor pair, such as a membrane, charge with a characteristic time ($\tau \equiv$ time constant) that is given by the product of the resistance and membrane ($\tau = RC$).

For membranes, $RC = \rho \frac{d}{A} \frac{\kappa}{4\pi k_e} \frac{A}{d} = \frac{\kappa \rho}{4\pi k_e} =$ constant for a given membrane.

In one time-constant, the capacitor (or membrane) reaches $(1 - e^{-1}) \times 100 \% = 63 \%$ of its final charge (or voltage).