This handout covers current and voltage changes in a single-loop circuit with a resistor and a capacitor. It provides a good model for a cell membrane, whose wall serves as both resistor and capacitor.

For the circuit above the initial charge of the capacitor is zero and the switch is closed at time $t = 0^+$. We have:

$$0 = -V_B + IR + \frac{Q}{C}.$$  

Recall that the current is just the change in charge with response to time, i.e., $I = \Delta Q/\Delta t$. This charge must end up on the capacitor plates. So:

$$0 = -V_B + \frac{\Delta Q}{\Delta t} R + \frac{Q}{C}.$$  

A little rearranging to place each term in the form of a current gives:

$$\frac{V_B}{R} = \frac{\Delta Q}{\Delta t} + \frac{Q}{\tau}.$$  

where $\tau = RC$. To solve this equation, we take the limit that $\Delta t \to 0$ and write:

$$\frac{V_B}{R} = \frac{dQ}{dt} + \frac{Q}{\tau}.$$  

The solution of this equation - recall that $\int_a^b \frac{dx}{x} = \log_e(b) - \log_e(a)$ and - for fun - try to solve it yourself - is:

$$Q(t) = CV_B \left(1 - e^{-t/\tau}\right).$$  

Thus $I = dQ/dt$ is:

$$I(t) = \frac{V_B}{R} e^{-t/\tau}$$  

and $I(t = 0^+) = V_B/R$ and $I(t \to \infty) = 0$. With $V = Q/C$, the voltage across the capacitor is:

$$V(t) = V_B \left(1 - e^{-t/\tau}\right).$$
and $V(t = 0^-) = 0$ and $V(t \to \infty) = V_B$.

A final point concerns the energy stored in the capacitor. The instantaneous power is just:

$$P(t) = I(t) V(t)$$

$$= \frac{V_B^2}{R} e^{-t/\tau} (1 - e^{-t/\tau})$$

$$= \frac{V_B^2}{R} (e^{-t/\tau} - e^{-2t/\tau})$$

The energy stored after the capacitor charges is the integral of the power, i.e.,

$$E = \int_0^\infty dt \, P(t)$$

This becomes - recall that $\int_a^b dx \, e^{\lambda x} = \frac{e^{b \lambda} - e^{a \lambda}}{\lambda}$ and - for fun - try to do it yourself! -

$$E = \frac{V_B^2}{R} \int_0^\infty (e^{-t/\tau} - e^{-2t/\tau})$$

$$= \frac{V_B^2}{R} \left( \frac{e^{-\infty} - e^{0}}{-1/\tau} - \frac{e^{-\infty} - e^{0}}{-2/\tau} \right)$$

$$= \frac{V_B^2}{R} \left( \frac{0-1}{-1/RC} - \frac{0-1}{-2/RC} \right)$$

$$= \frac{V_B^2}{R} \left( RC - \frac{RC}{2} \right)$$

$$= \frac{1}{2} C \frac{V_B^2}{2}$$

which is the expected result for energy in a capacitor.

**Bottom Line:** A resistor/capacitor pair, such as a membrane, charge with a characteristic time ($\tau \equiv$ time constant) that is given by the product of the resistance and membrane ($\tau = RC$).

For membranes, $RC = \rho \frac{\kappa}{A} \frac{A}{4\pi k_e d} = \frac{\kappa \rho}{4\pi k_e}$ = constant for a given membrane.

In one time-constant, the capacitor (or membrane) reaches $(1 - e^{-1}) \times 100\% = 63\%$ of its final charge (or voltage).