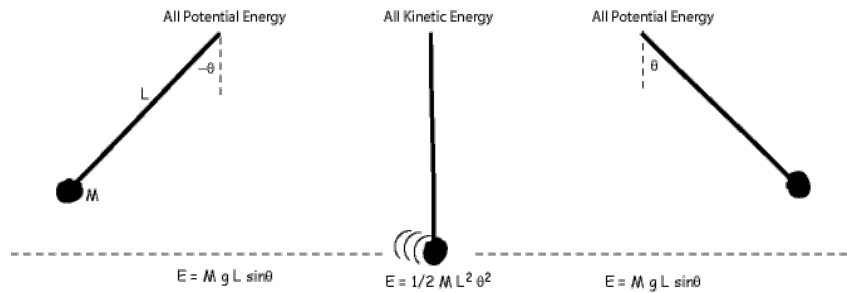
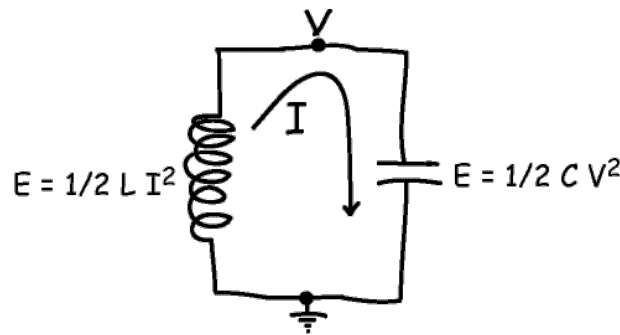


This handout covers the phenomena of resonance. This occurs when energy can alternate between one of two forms.

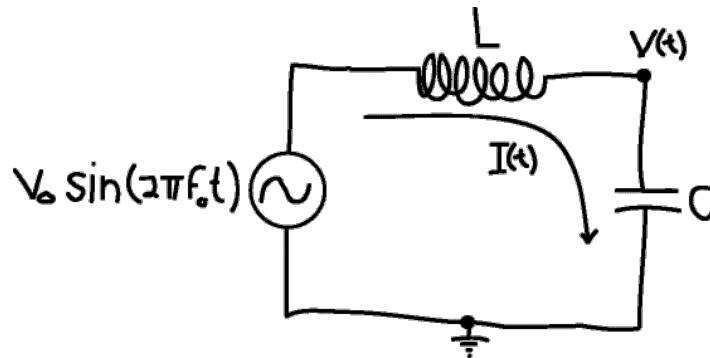
In a pendulum, or a mass dandling from a spring, the two forms are kinetic energy and potential energy. For example, in a pendulum the energy is all kinetics when the arm of the pendulum is at the lowest point of its arc and moving the fastest, while it is all kinetic when the arm is at the high point of the arc and instantaneously immobile.



In a LC circuit, the two forms are electrical potential energy, stored as the charge across a capacitor, and magnetic potential energy, stored as flux within a coil. In the absence of loss, i.e., with no resistance in the circuit, the energy sloses back and forth between the capacitor and the inductor.



Let's consider the dynamics in detail for a sinusoidal drive voltage,  $V_{drive} = V_0 \sin(2\pi f_0 t)$ .



Recall that the voltage drop across an inductor is:

$$V(t) = L \frac{dI(t)}{dt}$$

so that the current through the inductor is just:

$$I(t) = \frac{1}{L} \int_0^t dt' V(t') .$$

On the other hand, the current flow across a capacitor is:

$$I(t) = C \frac{dV(t)}{dt} .$$

If we apply Kirchoff's Law for current flow to the node labeled  $V(t)$ , we get:

$$\frac{1}{L} \int_0^t dt' [V(t') - V_0 \sin(2\pi f_0 t')] + C \frac{dV(t)}{dt} = 0 .$$

Thus:

$$\int_0^t dt' V(t') + LC \frac{dV(t)}{dt} = V_0 \frac{1}{2\pi f_0} \cos(2\pi f_0 t)$$

or, differentiating once:

$$V(t) + LC \frac{d^2V(t)}{dt^2} = -V_0 \sin(2\pi f_0 t) .$$

Let's stare at this. One steady-state solution is  $V(t) = A \sin(2\pi f t)$ , which gives

$$A \sin(2\pi f t) - A LC (2\pi f)^2 \sin(2\pi f t) = -V_0 \sin(2\pi f_0 t) .$$

This is a valid solution so long as:

$$f = f_0$$

and  $A - A LC (2\pi f_0)^2 = -V_0$ , or:

$$|A| = V_0 \frac{1}{1 - LC (2\pi f_0)^2} .$$

This demonstrates resonance, i.e., the amplitude explodes when  $LC = 1/(2\pi f_0)^2$ . We identify:

$$f_{\text{natural}} = \frac{1}{2\pi\sqrt{LC}}$$

as the natural frequency of this system - much like  $f_{\text{natural}} = \sqrt{g/l}$  is the natural frequency of a pendulum. Thus the current in the circuit,  $I(t) = C dV(t)/dt$ , is:

$$I(t) = - \frac{V_0 2\pi f_0 C}{1 - (f_0/f_{\text{natural}})^2} \cos(2\pi f_0 t) .$$

The change in sign, from "-" when  $f_0 < f_{\text{natural}}$  to "+" when  $f_0 > f_{\text{natural}}$ , corresponds to current leading versus lagging the voltage.