**Physics 2BL - David Kleinfeld**

# NOTES ON MEASURING THE RADIUS OF THE EARTH

We consider the sunset as viewed from two different heights, denoted $h_b$ and $h_c$ above the Earth's surface (figure). Our goal is to use the difference in height and the difference in sunset times, $t_b$ and $t_c$ respectfully, to calculate the radius of the earth, $R_E$. This involves calculating the time for the earth to rotate so that the tangents from each height to the sun (the definition of sunset) coincide.

We note that $h_b << R_E$ and $h_c << R_E$.

The angle that the earth rotates between the observation of sunset at the two heights is just

$$\Delta\cos(\theta) \equiv \cos(\theta_c) - \cos(\theta_b) = \frac{2\pi}{T}(t_c - t_b) \tag{1}$$

where $T$ is the period of the Earth's rotation ("1 day"). Thus

$$t_c - t_b = \frac{T}{2\pi}\left[\cos(\theta_c) - \cos(\theta_b)\right] \tag{2}$$

We now consider an expression for the angles $\theta_b$ and $\theta_c$, that is valid for measurements performed at the equator (and we are not at the equator, but the geometric correction is given below)

$$\cos(\theta_b) = \frac{R_E}{R_E + h_b} = \frac{1}{1 + \frac{h_b}{R_E}} \simeq 1 - \frac{h_b}{R_E} \tag{3}$$

But when $\cos(\theta_b) \simeq 1$, we can use the Taylor expansion to write

$$\cos(\theta_b) \simeq 1 - \frac{\theta^2}{2} \tag{4}$$

Thus

$$1 - \frac{\theta_b^2}{2} \simeq 1 - \frac{h_b}{R_E} \tag{5}$$

or

$$\theta_b \simeq \sqrt{\frac{2h_b}{R_E}} \tag{6}$$

Similarly,

$$\theta_c \simeq \sqrt{\frac{2h_c}{R_E}} \tag{7}$$

Combining terms gives

$$t_c - t_b = \frac{T}{2\pi}\left[\cos(\theta_c) - \cos(\theta_b)\right] = \frac{T}{2\pi}\frac{\sqrt{2h_c} - \sqrt{2h_b}}{\sqrt{R_E}} \tag{8}$$

This is rearranged to give

$$R_E = \frac{T^2}{2\pi^2}\left(\frac{\sqrt{h_c} - \sqrt{h_b}}{t_c - t_b}\right)^2 \tag{9}$$

We add a geometrical factor, denoted "$C$", to account for the latitude of La Jolla, denoted "$\lambda_{LJ}$", and for the tilt of the Earth's rotation relative to the orbital plane around the sun, denoted "$\lambda_{tilt}$".

$$R_E = \frac{1}{C}\frac{T^2}{2\pi^2}\left(\frac{\sqrt{h_c} - \sqrt{h_b}}{t_c - t_b}\right)^2 \tag{10}$$

where

$$C = \cos^2\lambda_{LJ}\cos^2\lambda_{tilt} - \sin^2\lambda_{LJ}\sin^2\lambda_{tilt} \tag{11}$$

with

$$\lambda_{LJ} = 32.87^o \tag{12}$$

and

$$\lambda_{tilt} = -23.4^o \sin\left(\frac{2\pi d}{365}\right) \tag{13}$$

where d is the number of days since 22 September or 20 March.

Lastly, we consider the issue of error analysis. There are three measured quantities in the expression for $R_E$ so there are three main contributions to the error in $R_E$, denoted as $\sigma_{R_E}$. These three terms, which must be summed in quadrature to obtain the final error, are

$$\left|\frac{\partial R_E}{\partial \Delta t}\right| \sigma_{\Delta t} = \frac{2R_E}{\Delta t}\sigma_{\Delta t} = \frac{4\pi}{T\left(\sqrt{h_c} - \sqrt{h_b}\right)}\sqrt{\frac{CR_E}{2}}\ \sigma_{\Delta t} \tag{14}$$

and

$$\left|\frac{\partial R_E}{\partial h_c}\right| \sigma_{h_c} = \frac{R_E}{\sqrt{h_c}\left(\sqrt{h_c} - \sqrt{h_b}\right)}\sigma_{h_c} \tag{15}$$

and

$$\left|\frac{\partial R_E}{\partial h_b}\right| \sigma_{h_b} = \frac{R_E}{\sqrt{h_b}\left(\sqrt{h_c} - \sqrt{h_b}\right)}\sigma_{h_b} \tag{16}$$

so that

$$\sigma_{R_E} = \sqrt{\left|\frac{\partial R_E}{\partial \Delta t}\right|^2 \sigma_{\Delta t}^2 + \left|\frac{\partial R_E}{\partial h_c}\right|^2 \sigma_{h_c}^2 + \left|\frac{\partial R_E}{\partial h_b}\right|^2 \sigma_{h_b}^2} \tag{17}$$