

NOTES ON MEASURING THE THICKNESS OF A SHELL

This experiment requires that you determine unknown physical parameters with nondestructive measurements. The fun part is that it requires that you synthesize a broad base of knowledge of mechanics. **The goal of the first part is to determine, nondestructively, the thickness of the skin of a hollow (or presumed hollow) ball of outer radius R_1 and inner radius R_2 .**

The outer radius is easily measured. The inner radius can be deduced in a number of ways. Lets sketch one scenario. Rotational motion may be used to deduce R_2 from the moment of inertia, I , of the ball. In particular, I is a function of R_1 and R_2 . We can deduce I from a measurement of the speed of a ball as it rolls down an inclined rail. Recall that the final speed depends on I , that is, on the distribution of the mass. The higher the moment the smaller the speed (which is one reason why some folks pay a small fortune to have very light rims and tires on their bicycles).

0.0.1 Moment of inertia, I

The moment of inertia is proportional to the total mass and is a function of the geometry of an object. For a solid ball spinning on one axis, we first calculate the moment of a disk about the axis, a height z above the origin. We than add all disks.

We recall that

$$dI = r^2 dm \tag{1}$$

where dm is a differential mass unit. For an integral over a region with uniform density, $dm = \rho dV$, where ρ is the density. Thus for a disk of radius R and thickness dz , integration in cylindrical coordinates gives

$$\begin{aligned} I_{disk} &= \int_0^R dr \int_{-\pi}^{\pi} r d\theta r^2 \rho dz \\ &= \rho dz \int_0^R r^3 dr \int_{-\pi}^{\pi} d\theta \\ &= \rho dz \frac{R^4}{4} 2\pi \\ &= \frac{\pi}{2} \rho R^4 dz \end{aligned} \tag{2}$$

We sum the contributions from each cylinder to calculate the moment for the solid ball of radius R_1 , i.e.,

$$\begin{aligned}
 I_{solid\ ball} &= \int_{-R_1}^{R_1} I_{disk} dz & (3) \\
 &= \frac{\pi}{2} \rho \int_{-R_1}^{R_1} R^4 dz \\
 &= 2 \frac{\pi}{2} \rho \int_0^{R_1} R^4 dz
 \end{aligned}$$

We further recall that for each disk of radius R ,

$$R_1^2 = R^2 + z^2 \quad (4)$$

Thus

$$\begin{aligned}
 I_{solid\ ball} &= \pi \rho \int_0^{R_1} \left(\sqrt{R_1^2 - z^2} \right)^4 dz & (5) \\
 &= \pi \rho \int_0^{R_1} (R_1^2 - z^2)^2 dz \\
 &= \pi \rho \int_0^{R_1} (R_1^4 - 2R_1^2 z^2 + z^4) dz \\
 &= \pi \rho \left(R_1^5 - \frac{2R_1^5}{3} + \frac{R_1^5}{5} \right) \\
 &= \pi \rho \frac{8R_1^5}{15}
 \end{aligned}$$

For a hollow ball, the moment of inertial is the same as that for a solid ball with R_2 subtracted from the ball with radius R_1 . Thus

$$I_{hollow\ ball} = \pi \rho \frac{8R_1^5}{15} - \pi \rho \frac{8R_2^5}{15} \quad (6)$$

$$= \pi \rho \frac{8}{15} (R_1^5 - R_2^5) \quad (7)$$

The density of the ball, which cannot be measured directly, may be expressed in terms of the total mass of the ball. Thus

$$\begin{aligned}
 M &= \int_V \rho & (8) \\
 &= \rho \int_{R_2}^{R_1} dr \int_{-\pi}^{\pi} r d\theta \int_{-\pi/2}^{\pi/2} r \sin \phi d\phi \\
 &= \rho \frac{4\pi}{3} (R_1^3 - R_2^3)
 \end{aligned}$$

or

$$\rho = M \frac{3}{4\pi} \frac{1}{R_1^3 - R_2^3} \quad (9)$$

Thus we finally get

$$\begin{aligned} I_{\text{hollow ball}} &= \pi M \frac{3}{4\pi} \frac{1}{R_1^3 - R_2^3} \frac{8}{15} (R_1^5 - R_2^5) \\ &= \frac{2}{5} M \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3} \\ &= \frac{2}{5} M R_1^2 \frac{1 - \left(\frac{R_2}{R_1}\right)^5}{1 - \left(\frac{R_2}{R_1}\right)^3} \end{aligned} \quad (10)$$

0.0.2 Kinematics - Via Energy Conservation

If we roll a ball down an inclined plane, so that it falls through a height H , the final speeds, V_f and ω_f , and initial speeds, V_i and ω_i , are related by:

$$MgH + \frac{1}{2}MV_i^2 + \frac{1}{2}I\omega_i^2 = \frac{1}{2}MV_f^2 + \frac{1}{2}I\omega_f^2 \quad (11)$$

We can take V_i to be zero, similarly for ω_i . Thus

$$\begin{aligned} MgH &= \frac{1}{2}MV_f^2 + \frac{1}{2}I\omega_f^2 \\ &= \frac{1}{2}MV_f^2 + \frac{1}{2} \cdot \frac{2}{5} M R_1^2 \frac{1 - \left(\frac{R_2}{R_1}\right)^5}{1 - \left(\frac{R_2}{R_1}\right)^3} \left(\frac{V_f}{R'}\right)^2 \\ &= \frac{1}{2}MV_f^2 \left[1 + \frac{2}{5} \left(\frac{R_1}{R'}\right)^2 \frac{1 - \left(\frac{R_2}{R_1}\right)^5}{1 - \left(\frac{R_2}{R_1}\right)^3} \right] \end{aligned} \quad (12)$$

where R' is the radius along which the ball rolls. For a ball that rolls on rails, $R' < R_1$.

The final velocity may be found from measuring the time it take the ball to roll a distance L along the channel as the ball drops in height by H (of course, H and L are related by the slope of the channel). When the

acceleration is constant, as it is for the case of gravity at the surface of the Earth, the height is given by

$$\begin{aligned} L &= \frac{1}{2}(V_f + V_i)T \\ &= \frac{V_f}{2}T \end{aligned} \quad (13)$$

where T is the measured time to fall through H . Thus the final speed is

$$V_f = \frac{2L}{T} \quad (14)$$

0.0.3 The Main Equation

We now put all the pieces together to get

$$\begin{aligned} MgH &= \frac{1}{2}MV_f^2 \left[1 + \frac{2}{5} \left(\frac{R_1}{R'} \right)^2 \frac{1 - \left(\frac{R_2}{R_1} \right)^5}{1 - \left(\frac{R_2}{R_1} \right)^3} \right] \\ &= \frac{1}{2}M \left(\frac{2L}{T} \right)^2 \left[1 + \frac{2}{5} \left(\frac{R_1}{R'} \right)^2 \frac{1 - \left(\frac{R_2}{R_1} \right)^5}{1 - \left(\frac{R_2}{R_1} \right)^3} \right] \end{aligned} \quad (15)$$

The mass term, M , will cancel as it appears linearly in all terms. We find

$$\frac{gHT^2}{2L^2} = \left[1 + \frac{2}{5} \left(\frac{R_1}{R'} \right)^2 \frac{1 - \left(\frac{R_2}{R_1} \right)^5}{1 - \left(\frac{R_2}{R_1} \right)^3} \right] \quad (16)$$

or

$$\frac{5}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) = \frac{1 - \left(\frac{R_2}{R_1} \right)^5}{1 - \left(\frac{R_2}{R_1} \right)^3} \quad (17)$$

This is a transcendental equation, that may be solved graphically or numerically for $\frac{R_2}{R_1}$ in terms of the value of the experimentally determined quantities on the left hand side. It is in the form of

$$C = \frac{1 - x^5}{1 - x^3} \quad (18)$$

where C consists of experimentally measured quantities, i.e., $C = \frac{5}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right)$ and $x \equiv \frac{R_2}{R_1}$. If one plots $C(1-x^3)$ versus x and $1-x^5$ versus x , the solutions is given by the intersection of these two curves. Likewise, one can plot $C = \text{constant}$ versus $\frac{1-x^5}{1-x^3}$, as shown on the attached figure.

The experimentally determined value of the ratio $\frac{R_2}{R_1}$ is used to deduce R_2 , and thus the thickness, Δ , in terms of measurable quantities, i.e.,

$$\begin{aligned} \Delta &\equiv R_1 - R_2 = R_1 \left(1 - \frac{R_2}{R_1} \right) \\ &= R_1 (1 - x) \end{aligned} \quad (19)$$

0.0.4 Approximate Form for Error Analysis

The expression for $\frac{R_2}{R_1}$ is a fifth-order equation and thus does not have a closed form solution. This makes it difficult - but not impossible - to calculate the partial derivatives for the error analysis. We can get an approximate relation for the thickness by rewriting the above result in terms of $R_2 = R_1 - \Delta$, i.e.,:

$$\begin{aligned} \frac{5}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) &= \frac{1 - \left(\frac{R_1 - \Delta}{R_1} \right)^5}{1 - \left(\frac{R_1 - \Delta}{R_1} \right)^3} \\ &= \frac{1 - \left(1 - \frac{\Delta}{R_1} \right)^5}{1 - \left(1 - \frac{\Delta}{R_1} \right)^3} \\ &= \frac{1 - \left(1 - 5 \left(\frac{\Delta}{R_1} \right) + 10 \left(\frac{\Delta}{R_1} \right)^2 - \dots \right)}{1 - \left(1 - 3 \left(\frac{\Delta}{R_1} \right) + 3 \left(\frac{\Delta}{R_1} \right)^2 - \dots \right)} \\ &= \frac{5 \left(\frac{\Delta}{R_1} \right) - 2 \left(\frac{\Delta}{R_1} \right)^2 + \dots}{3 \left(\frac{\Delta}{R_1} \right) - \left(\frac{\Delta}{R_1} \right)^2 + \dots} \\ &= \frac{5}{3} \frac{1 - 2 \left(\frac{\Delta}{R_1} \right) + \dots}{1 - \left(\frac{\Delta}{R_1} \right) + \dots} \\ &= \frac{5}{3} \left(1 - 2 \frac{\Delta}{R_1} + \dots \right) \left(1 + \frac{\Delta}{R_1} - \dots \right) \\ &= \frac{5}{3} \left(1 - \frac{\Delta}{R_1} + \dots \right) \end{aligned} \quad (20)$$

For completeness, this is in the form

$$C = \frac{5}{3}x \quad (21)$$

and is valid when the value of x is close to $x=1$. This is shown on the attached figure. Getting back to the approximate result, we note that to *first order* approximation

$$\frac{3}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) \simeq 1 - \frac{\Delta}{R_1} \quad (22)$$

or

$$\Delta \simeq R_1 \left[1 - \frac{3}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) \right] \quad (23)$$

The above expression is useful for estimating the partial derivative terms that go into the error estimate for Δ , but is too crude to determine Δ without solution of the exact expression for Δ . We find that the five error terms that must be added in quadrature are given by:

$$\left| \frac{\partial \Delta}{\partial L} \right| \delta L \simeq \frac{3}{2} R_1 \left(\frac{R'}{R_1} \right)^2 \frac{gHT^2}{L^2} \frac{\delta L}{L} \quad (24)$$

$$\left| \frac{\partial \Delta}{\partial T} \right| \delta T \simeq \frac{3}{2} R_1 \left(\frac{R'}{R_1} \right)^2 \frac{gHT^2}{L^2} \frac{\delta T}{T} \quad (25)$$

$$\left| \frac{\partial \Delta}{\partial H} \right| \delta H \simeq \frac{3}{4} R_1 \left(\frac{R'}{R_1} \right)^2 \frac{gHT^2}{L^2} \frac{\delta H}{H} \quad (26)$$

$$\left| \frac{\partial \Delta}{\partial R'} \right| \delta R' \simeq 3 R_1 \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) \frac{\delta R'}{R'} \quad (27)$$

$$\left| \frac{\partial \Delta}{\partial R_1} \right| \delta R_1 \simeq R_1 \sqrt{1 + \left[\frac{3}{2} \left(\frac{R'}{R_1} \right)^2 \left(\frac{1}{2} \frac{gHT^2}{L^2} - 1 \right) \right]^2} \frac{\delta R_1}{R_1} \quad (28)$$

These formula are not very complicated when one groups parameters in terms of dimensionless quantities, i.e, $\left(\frac{R'}{R_1} \right)^2$ and $\frac{gHT^2}{L^2}$, as done above. The total estimated error is

$$\delta\Delta \simeq \tag{29}$$

$$\sqrt{\left(\left|\frac{\partial\Delta}{\partial T}\right|\delta T\right)^2 + \left(\left|\frac{\partial\Delta}{\partial H}\right|\delta H\right)^2 + \left(\left|\frac{\partial\Delta}{\partial L}\right|\delta L\right)^2 + \left(\left|\frac{\partial\Delta}{\partial R'}\right|\delta R'\right)^2 + \left(\left|\frac{\partial\Delta}{\partial R_1}\right|\delta R_1\right)^2}$$

There are other, equivalent and equally valid ways to estimate the error, such as graphically or numerically in terms of the full equation for the kinematics.

