Notes on the Critically Damped Harmonic Oscillator

Physics 2BL - David Kleinfeld

We often have to build an electrical or mechanical device. An understanding of physics may help in the design and tuning of such a device. Here, we consider a critically damped spring oscillator as a model design for the shock absorber of a car.

We consider a mass, denoted m, that is connected to a spring with spring constant k, so that the restoring force is $F = -kx$, and which moves in a lossy manner so that the frictional force is $F = -bv = -b\dot{x}$. Prof. Newton tells us that

$$\sum F = m\ddot{x} = -kx - b\dot{x} \tag{1}$$

Thus

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0 \tag{2}$$

The two reduced constants are the natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3}$$

and the decay constant

$$\alpha = \frac{b}{m} \tag{4}$$

so that we need to consider

$$\ddot{x} + \omega_0^2 x + \alpha\dot{x} = 0 \tag{5}$$

The above equation describes simple harmonic motion with loss. It is discussed in lots of text books, but I want to consider a formulation of the solution that is most natural for critical damping.

We know that when the damping constant is zero, i.e., $\alpha = 0$, the solution of $\ddot{x} + \omega_0^2 x = 0$ is given by:

$$x(t) = Ae^{+i\omega_0 t} + Be^{-i\omega_0 t} \tag{6}$$

where A and B are constants that are found from the initial conditions, i.e., $x(0)$ and $\dot{x}(0)$. In a nut shell, the system oscillates forever.

We know that when the the natural frequency is zero, i.e., $\omega_0 = 0$, the solution of $\ddot{x} + \alpha\dot{x} = 0$ is given by:

$$\dot{x}(t) = Ae^{-\alpha t} \tag{7}$$

and

$$x(t) = A\frac{1 - e^{-\alpha t}}{\alpha} + B \tag{8}$$

where A and B are constants that are found from the initial conditions. In a nut shell, the system grinds to a halt.

A parenthetical remark, of relevance in the laboratory exercises, is that in the presence of a constant force field, like gravity, the main equation becomes $\ddot{x} + \alpha\dot{x} + g = 0$ and the solution simply picks up a constant to become

$$\dot{x}(t) = Ae^{-\alpha t} - \frac{g}{\alpha} \tag{9}$$

In a nut shell, the system reaches a terminal velocity of

$$\dot{x}(t \leftarrow \infty) = \frac{mg}{b} \tag{10}$$

on the time-scale of $t >> \alpha^{-1}$.

To return to the general case, we see that the presence of a decay term leads to an exponential loss in the amplitude of the system. It is that natural to suppose that the damped oscillator has a solution of the form

$$x(t) = e^{-\beta t}u(t) \tag{11}$$

where β is a constant and we suspect that $u(t)$ may be the solution to an undamped harmonic oscillator. We can test this idea by computing derivatives and substituting them back into the original equation. We have

$$\begin{aligned} \dot{x}(t) &= -\beta e^{-\beta t}u(t) + e^{-\beta t}\dot{u}(t) \\ &= e^{-\beta t}\left[\dot{u}(t) - \beta u(t)\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} \ddot{x}(t) &= -\beta e^{-\beta t}\left(\dot{u}(t) - \beta u(t)\right] + e^{-\beta t}\left[\ddot{u}(t) - \beta\dot{u}(t)\right] \\ &= e^{-\beta t}\left[\ddot{u}(t) - 2\beta\dot{u}(t) + \beta^2 u(t)\right] \end{aligned} \tag{13}$$

Thus

$$e^{-\beta t}\left[\ddot{u}(t)-2\beta\dot{u}(t)+\beta^2u(t)+\omega_0^2u(t)+\alpha\dot{u}(t)-\alpha\beta u(t)\right]=0. \tag{14}$$

Since the prefactor $e^{-\beta t}$ is never zero, the term in the brackets must be zero. This term simplifies considerably when the factors in front of the $\dot{u}(t)$ terms sum to zero, which occurs for the choice

$$\beta=\frac{\alpha}{2} \tag{15}$$

Then we have

$$\ddot{u}(t)+\left[\omega_0^2-\left(\frac{\alpha}{2}\right)^2\right]u(t)=0 \tag{16}$$

which is the equation for simple harmonic motion with a frequency given by

$$\omega=\sqrt{\omega_0^2-\left(\frac{\alpha}{2}\right)^2} \tag{17}$$

so that for $\omega\neq 0$

$$u(t)=Ae^{+i\omega t}+Be^{-i\omega t} \tag{18}$$

or

$$x(t)=e^{-\frac{\alpha}{2}t}\left[Ae^{+i\sqrt{\omega_0^2-\left(\frac{\alpha}{2}\right)^2}\;t}+Be^{-i\sqrt{\omega_0^2-\left(\frac{\alpha}{2}\right)^2}\;t}\right] \tag{19}$$

When $\omega_0>\frac{\alpha}{2}$, ω is real and the solution has an oscillatory component. This is called the underdamped solution. When $\omega_0<\frac{\alpha}{2}$, ω is imaginary and the solution is an exponential decay. This is called the overdamped solution.

The interesting case for us is when $\omega_0=\frac{\alpha}{2}$, so that

$$\ddot{u}(t)=0 \tag{20}$$

The solution is

$$u(t)=A+Bt \tag{21}$$

so that

$$x(t)=e^{-\frac{\alpha}{2}t}\left[A+Bt\right] \tag{22}$$

This is denoted critical damping. In terms of the initial conditions

$$x(t)=e^{-\frac{\alpha}{2}t}\left[x(0)\left(1+\frac{\alpha}{2}t\right)+\dot{x}(0)t\right]. \tag{23}$$

To simply matters in graphing $x(t)$, we take $\dot{x}(0) = 0$, so that

$$x(t) = x(0)e^{-\frac{\alpha}{2}t}\left(1 + \frac{\alpha}{2}t\right). \tag{24}$$

and

$$\dot{x}(t) = -x(0)\frac{\alpha}{2}e^{-\frac{\alpha}{2}t}\left(\frac{\alpha}{2}t\right) \tag{25}$$

and

$$\ddot{x}(t) = x(0)\left(\frac{\alpha}{2}\right)^2 e^{-\frac{\alpha}{2}t}\left(\frac{\alpha}{2}t - 1\right) \tag{26}$$

Before we set out to graph the above kinematic variables, we note that

$$x(t \leftarrow 0) = x(0) + O(t^2), \tag{27}$$

so that the slope of $x(t \leftarrow 0)$ is zero, i.e., $\dot{x}(0) = 0$. We also note that

$$\dot{x}(t \leftarrow 0) = -x(0)\frac{\alpha}{2}t + O(t^2) \tag{28}$$

and

$$\ddot{x}(t \leftarrow 0) = -x(0)\left(\frac{\alpha}{2}\right)^2 + O(t), \tag{29}$$

so that the system slows down with a constant deceleration from the very start. Lastly, we also see that the speed peaks at

$$t_{max} = \frac{2}{\alpha}. \tag{30}$$

In some sense, critical damping gives a "gentle" return to baseline.