UCSD — Physics 120 — Spring 2019 — Midterm Solutions

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16th May, 2019

This exam was scored out of **100** points. The average ± stdev was **56.4** ± **17.7** this quarter. The distribution of scores for the class is shown in Figure 0 below.

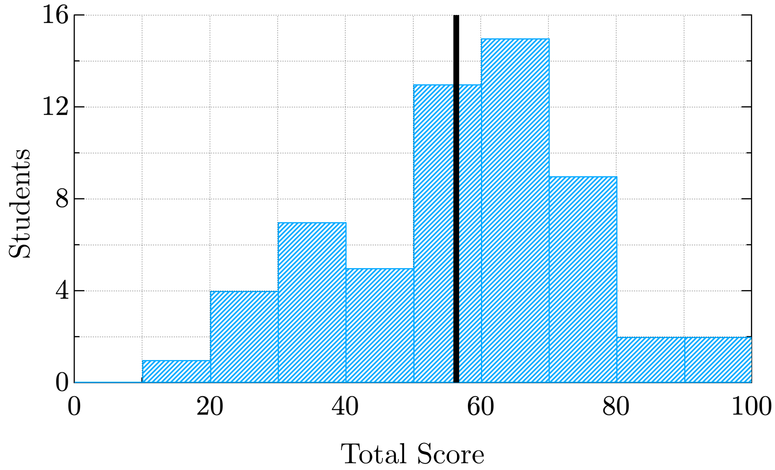

Figure 0: The grade distribution. The black bar indicates the average.

Problem 1. **34**

We have a resistor in series with an inductor and capacitor in parallel, with values R, L and C respectively. The question concerns the total impedance between either end of this circuit.

Problem 1a. **15**

These elements have impedances $Z_R = R$, $Z_L = i\omega L$ and $Z_C = \frac{1}{i\omega C}$. **5**

So, recalling that elements in series add linearly, and elements in parallel add inversely, **5**

$$\begin{aligned} \hat{Z}_{\text{eq}}(\omega) = R + (L||C) &= Z_R + \left(\frac{1}{Z_L} + \frac{1}{Z_C}\right)^{-1} \\ &= R + \left(\frac{1}{i\omega C} + i\omega L\right)^{-1}. \end{aligned} \tag{1}$$

So, after some simplification, **5**

$$\hat{Z}_{\text{eq}}(\omega) = R + \frac{i\omega L}{1 - \omega^2 LC}. \tag{2}$$

Problem 1b. 11

First, let's combine the fraction and divide by R,

$$\frac{\hat{Z}_{\rm eq}(\omega)}{R} = \frac{R - \omega^2 RLC + i\omega L}{1 - \omega^2 LC}. \tag{3}$$

The magnitude we're interested in is then given by

$$\begin{aligned} \left|\frac{\hat{Z}_{\rm eq}(\omega)}{R}\right|^2 &= \frac{1}{R^2}\hat{Z}^*_{\rm eq}\hat{Z}_{\rm eq} = \frac{1}{R^2}\frac{R - \omega^2 RLC + i\omega L}{1 - \omega^2 LC}\frac{R - \omega^2 RLC - i\omega L}{1 - \omega^2 LC} \\ &= \frac{\omega^2 \frac{L^2}{R^2} + \left(1 - \omega^2 LC\right)^2}{\left(1 - \omega^2 LC\right)^2}. \end{aligned} \tag{4}$$

Note then that $\frac{L}{R} \ll RC$ and thus the first term can be dropped, so finally 2

$$\left|\frac{\hat{Z}_{\rm eq}(\omega)}{R}\right|^2 = \frac{\left(1 - \omega^2 LC\right)^2}{\left(1 - \omega^2 LC\right)^2} \Rightarrow \left|\frac{\hat{Z}_{\rm eq}(\omega)}{R}\right| = \frac{1 - \omega^2 LC}{1 - \omega^2 LC}. \tag{5}$$

This is flat at zero dB except for a infinitely positive delta function at $\omega = \sqrt{LC}$. 3

Our hand-drawn Bode plot is shown in Figure 1 below. 6

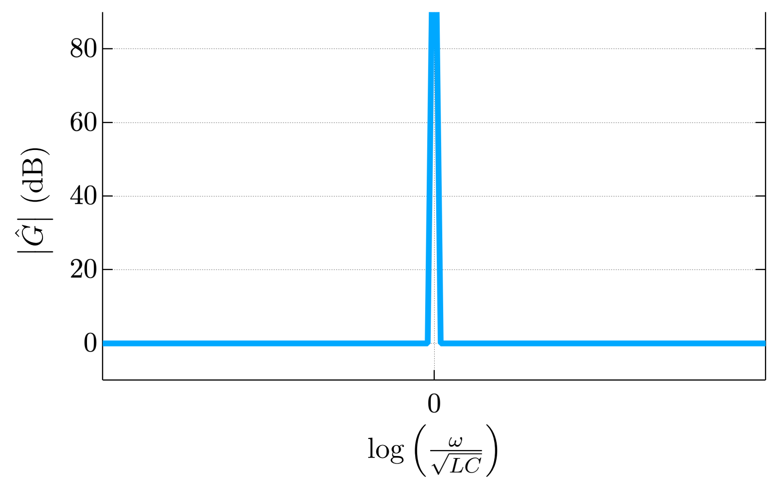

Figure 1: Bode Plot of $\left|\hat{G}(\omega)\right|$ for Problem **1b**.

Problem 1c. 8

The phase is given by 2

$$\phi = \arg\left(\frac{{\rm Im}[\hat{Z}_{\rm eq}]}{{\rm Re}[\hat{Z}_{\rm eq}]}\right) = \arctan\left(\frac{{\rm Im}[\hat{Z}_{\rm eq}]}{{\rm Re}[\hat{Z}_{\rm eq}]}\right). \tag{6}$$

As before, since the only imaginary part is controlled by a factor of $\frac{L}{R} \ll RC$, we know that $\phi \approx \arctan(0) = 0$ for all $\omega \neq \sqrt{LC}$. However, as ω approaches $\sqrt{LC}$ from below, we see that ϕ will approach $+\frac{\pi}{2}$. Conversely, as we approach from above, ϕ will approach $-\frac{\pi}{2}$. 2

So, we can construct a crude Bode plot from this, shown in Figure 2 below. 4

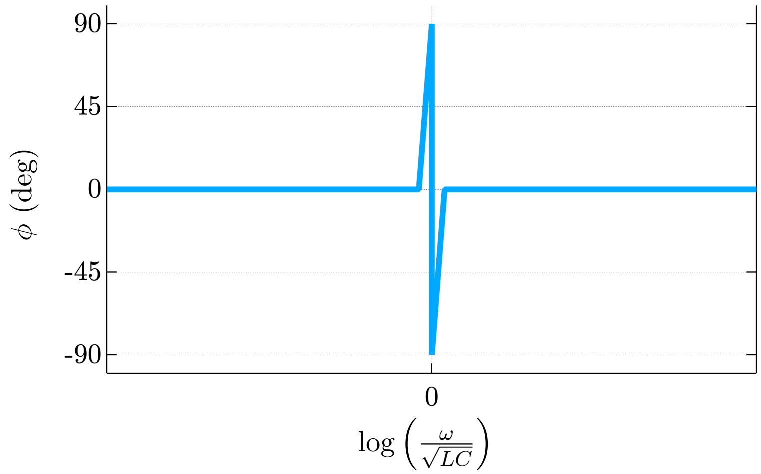

Figure 2: Bode Plot of $\phi(\omega)$ for Problem **1c**.

Problem 2. 31

We have an ideal op-amp with a negative feedback network; a resistor and inductor in series at the input, and a resistor and capacitor in parallel as feedback. The question concerns the complex gain of this circuit.

Problem 2a. 15

We'll start with our favorite op-amp equation, 1

$$V_{\text{out}} = A\left(V_+ - V_-\right). \tag{7}$$

Seeing that $V_+ = 0$ and substituting, 1

$$V_{\text{out}} = -AV_-. \tag{8}$$

Now, we can match currents, 6

$$\frac{V_{\text{in}} - 0}{R + i\omega L} = \frac{0 - V_{\text{out}}}{\left(\frac{1}{R} + i\omega C\right)^{-1}}, \tag{9}$$

and solve for the complex gain in the limit that $A \to \infty$, 6

$$\hat{G} \equiv \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{\left(\frac{1}{R} + i\omega C\right)^{-1}}{i\omega L + R} = -\frac{R}{\left(1 + i\omega RC\right)\left(R + i\omega L\right)}. \tag{10}$$

Problem 2b. 16

Let's take the magnitude of $\hat{G}$,

$$\begin{aligned} |\hat{G}| = \sqrt{GG^*} &= \sqrt{\frac{R}{(1+i\omega RC)(R+i\omega L)} \frac{R}{(1-i\omega RC)(R-i\omega L)}} \\ &= \sqrt{\frac{R^2}{(R^2+\omega^2 L^2)(1+\omega^2 R^2 C^2)}}. \end{aligned} \tag{11}$$

In standard form,

$$|\hat{G}| = \left(1 - \frac{L^2}{R^2}\omega^2\right)^{-\frac{1}{2}} \left(1 - R^2C^2\omega^2\right)^{-\frac{1}{2}}. \tag{12}$$

Given that $RC = 10\frac{L}{R}$,

$$|\hat{G}| = \left(1 - \frac{L^2}{R^2}\omega^2\right)^{-\frac{1}{2}} \left(1 - 100\frac{L^2}{R^2}\omega^2\right)^{-\frac{1}{2}}. \tag{13}$$

There are two poles here, both first order, $\omega_1 = \frac{L}{R}$, and $\omega_2 = 10\frac{L}{R}$. 4

So, putting this on our Bode plot as discussed in the review session, we'll have -20dB per decade after the first pole, and they'll stack to -40dB per decade afterwards. 4

As for the asymptotes, we don't have a constant in front, so we start at 0dB, and the gain drops to zero (-∞dB) as ω grows. 4

Finally, our hand-drawn Bode plot is shown in Figure 3 below. 4

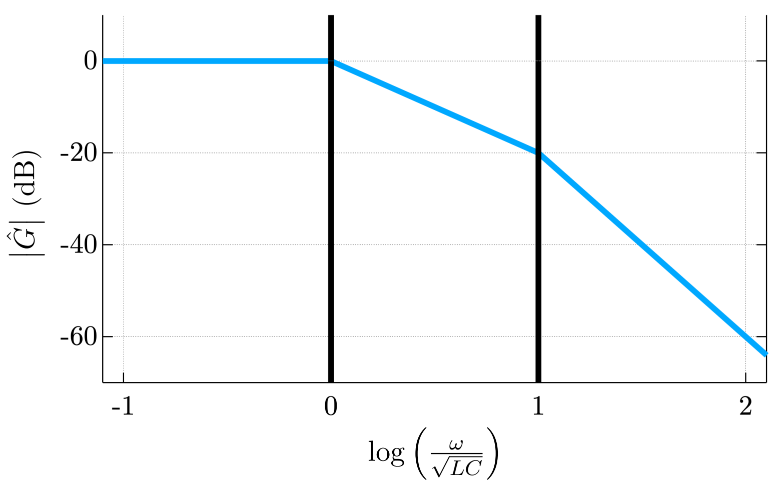

Figure 3: Bode Plot of $\left|\hat{G}(\omega)\right|$ for Problem **2b**. The black bars indicate break frequencies.

Problem 3. 20

We have a diode clamp, with a resistor to prevent shorting the diode.

For our ideal diode, if $V_{\text{diode}} \geq 0.6\text{V}$, then the diode is on and fully conductive; otherwise, it does not conduct at all. 5

So, since $V_{\text{diode}} = V_{\text{out}}$, any time $V_{\text{out}} \geq 0.6\text{V}$, the diode will conduct and V_{out} will be clamped to 0.6V. As such, our voltages are as shown in Figure 4 below. 15

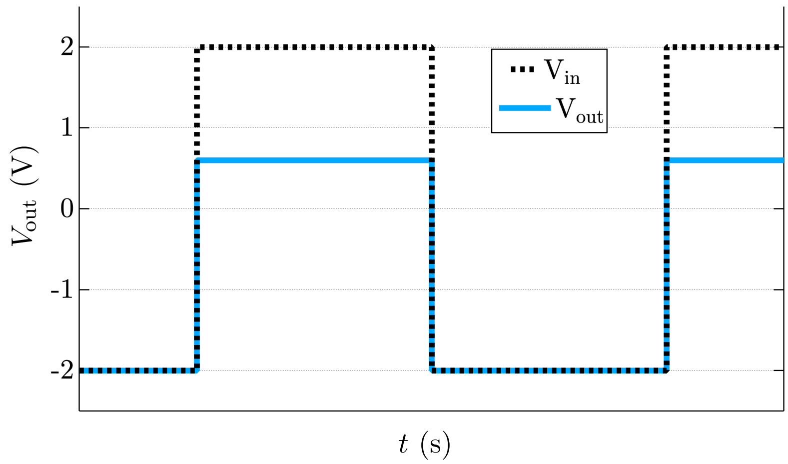

Figure 4: Plot of V_{in} and V_{out} for Problem **3**.

Problem 4. 15

We have a constant function, and want to find the Fourier transform.

To take the Fourier transform of $f(t) = 1$, we apply the formula on the fifth line of the front of the formula sheet, 5

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} 1e^{-i\omega t}dt. \end{aligned} \tag{14}$$

Using the second-to-last equation on the back fo the formula sheet, 10

$$\begin{aligned} \delta(x - x_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy e^{-i(x-x_0)y} \\ &\Rightarrow F(\omega) = 2\pi\delta(\omega). \end{aligned} \tag{15}$$