### *Power in reactive circuits*

The instantaneous power delivered to any circuit element is always given by the product $P = VI$. However, in reactive circuits where $V$ and $I$ are not simply proportional, you can't just multiply them together. Funny things can happen; for instance, the sign of the product can reverse over one cycle of the ac signal. Figure 1.49 shows an example. During time intervals $A$ and $C$, power is being delivered to the capacitor (albeit at a variable rate), causing it to charge up; its stored energy is increasing (power is the rate of change of energy). During intervals $B$ and $D$, the power delivered to the capacitor is negative; it is discharging. The average power over a whole cycle for this example is in fact exactly zero, a statement that is always true for any purely reactive circuit element (inductors, capacitors, or any combination thereof). If you know your trigonometric integrals, the next exercise will show you how to prove this.

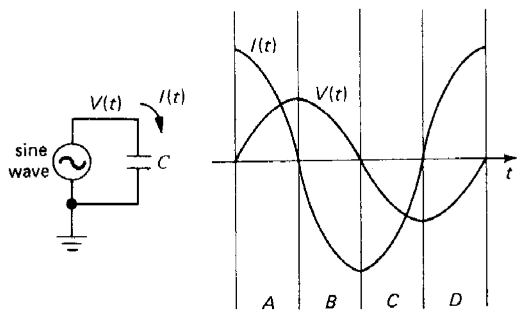


Figure 1.49. When driven by a sine wave, the current through a capacitor leads the voltage by 90°.

EXERCISE 1.18

Optional exercise: Prove that a circuit whose current is 90° out of phase with the driving voltage consumes no power, averaged over an entire cycle.

How do we find the average power consumed by an arbitrary circuit? In general, we can imagine adding up little pieces of $VI$ product, then dividing by the elapsed time. In other words,

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt$$

where $T$ is the time for one complete cycle. Luckily, that's almost never necessary. Instead, it is easy to show that the average power is given by

$$P = \mathcal{R}e(\mathbf{VI}^*) = \mathcal{R}e(\mathbf{V}^*\mathbf{I})$$

where **V** and **I** are complex rms amplitudes.

Let's take an example. Consider the preceding circuit, with a 1 volt (rms) sine wave driving a capacitor. We'll do everything with rms amplitudes, for simplicity. We have

$$\mathbf{V} = 1$$

$$\mathbf{I} = \frac{\mathbf{V}}{-j/\omega C} = j\omega C$$

$$P = \mathcal{R}e(\mathbf{VI}^*) = \mathcal{R}e(-j\omega C) = 0$$

That is, the average power is zero, as stated earlier.

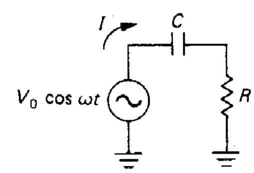


Figure 1.50

As another example, consider the circuit shown in Figure 1.50. Our calculations go like this:

$$\mathbf{Z} = R - \frac{j}{\omega C}$$

$$\mathbf{V} = V_0$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0}{R - (j/\omega C)} = \frac{V_0[R + (j/\omega C)]}{R^2 + (1/\omega^2 C^2)}$$

$$P = \mathcal{R}e(\mathbf{VI}^*) = \frac{V_0^2 R}{R^2 + (1/\omega^2 C^2)}$$

(In the third line we multiplied numerator and denominator by the complex conjugate of the denominator, in order to make the denominator real.) This is less than the product of the magnitudes of **V** and **I**. In fact, the ratio is called the *power factor*:

$$|\mathbf{V}|\,|\mathbf{I}| = \frac{V_0^2}{[R^2 + (1/\omega^2 C^2)]^{1/2}}$$

$$\text{power factor} = \frac{\text{power}}{|\mathbf{V}|\,|\mathbf{I}|} = \frac{R}{[R^2 + (1/\omega^2 C^2)]^{1/2}}$$

in this case. The power factor is the cosine of the phase angle between the voltage and the current, and it ranges from 0 (purely reactive circuit) to 1 (purely resistive). A power factor less than 1 indicates some component of reactive current.

EXERCISE 1.19

Show that all the average power delivered to the preceding circuit winds up in the resistor. Do this by computing the value of $V_R^2/R$. What is that power, in watts, for a series circuit of a 1$\mu$F capacitor and a 1.0k resistor placed across the 110 volt (rms), 60Hz power line?

Power factor is a serious matter in large-scale electrical power distribution, because reactive currents don't result in useful power being delivered to the load, but cost the power company plenty in terms of $I^2R$ heating in the resistance of generators, transformers, and wiring. Although residential users are only billed for "real" power [$\mathcal{R}e(\mathbf{VI}^*)$], the power company charges industrial users according to the power factor. This explains the capacitor yards that you see behind large factories, built to cancel the inductive reactance of industrial machinery (i.e., motors).

EXERCISE 1.20

Show that adding a series capacitor of value $C = 1/\omega^2 L$ makes the power factor equal 1.0 in a series $RL$ circuit. Now do the same thing, but with the word "series" changed to "parallel."