1 Basics of noise and signal-to-noise in electronics

Fluctuations in physical quantities are an essential aspect of our world. These arise because of sampling, such as when we flip a coin N times and note that the variance is N, or the standard deviation is $\sqrt{N}$. They also arise from thermal activation, such as the diffusive motion of a molecule in solution. We will consider two of these sources within the context of electrical circuits. We ignore the large zoo of so called "technical" noise sources that result from imperfections in materials (noting that defects themselves in solids result from thermal fluctuations).

1.1 Fluctuations from resistance

Lossy devices like resistors always have a noise associated with the random movement of charge carriers. This can be expressed in a general way in terms of the fluctuation-dissipation theorem that relates the loss in energy in a device to the level of fluctuation in the transport of a quantity, such a charge. Thus for the case of electrical circuits, an initial current through a closed loop that contains a resistor will rapidly go toward zero because of the resistance. This occurs because the resistance dissipates electrical energy, turning it into heat (Joule heating). However, there is a random fluctuating current flow through the resistor that is caused by the thermal fluctuations of the electrons in the resistor. This is called Johnson noise. It has zero mean and a variance δI_J^2 given by value of the current, is

$$\delta I_J^2 = \frac{4k_B T \Delta\nu}{R}. \tag{1.1}$$

A complementary view is to look at the voltage in an open circuit across the resistor (Figure 1). Here the variance of the voltage noise is

$$\delta V_J^2 = 4k_B T R \Delta\nu. \tag{1.2}$$

. Note that a resistor also has a nonzero capacitance between the leads, denoted C. The parallel combination of R and C will lead to a low pass filter with time constant RC. Thus the bandwidth will be limited to a maximum of $\Delta\nu \approx 1/RC$. A more precise estimate can be found by equating the decrement in power for a RC low-pass filter with $\Delta\nu$, *i.e.*,

$$\Delta\nu = \frac{1}{2\pi}\int_0^\infty d\omega \; \frac{1}{1+(\omega/RC)^2} = \frac{1}{4RC}. \tag{1.3}$$

Thus the variance of the voltage noise can also be expressed as

$$\delta V_J^2 = 4k_B T R \frac{1}{4RC} = \frac{k_B T}{C} \tag{1.4}$$

which is interesting, and a bit intuitive, as capacitance corresponds to a length scale, say L, so that the noise varies like one over the size of the system, *i.e.*, $\delta V_J^2 \propto 1/L$.

Another way to derive the expression for the noise voltage is through equipartition, where we equate the fluctuations in the energy stored in the capacitor with the thermal energy for one degree of freedom. Thus

$$\frac{1}{2}C\delta V_J^2 = \frac{1}{2}k_B T \tag{1.5}$$

and we again find

$$\delta V_J^2 = 4k_B T R \frac{1}{4RC} = \frac{k_B T}{C}. \tag{1.6}$$

How big is this? For a typical resistor, such as the ones we use in class, $C \approx 10$ picoFarads, so

$$\delta V^2 = \frac{k_B T}{e}\frac{e}{C} \approx 2.5x10^{-2}\text{Volts}\frac{1.6x10^{-19}\text{Coulombs}}{1x10^{-11}\text{Farads}} \approx 4x10^{-10}(\text{Volts})^2 \tag{1.7}$$

where e is the elementary electronic charge. The standard deviation, of root-mean-square value of the fluctuations, is $\delta V \approx 2x10^{-5}$ Volts, or 20 μV. This sets a scale for voltage noise added by a resistor.

1.2 Fluctuations from counting

A second source of noise is the shot effect. In electronics shot noise originates from the discrete nature of electric charge. Shot noise also occurs in photon counting in optical devices, where shot noise is associated with the particle nature of light. If we measure N photons, we expect that number to vary by $\delta N^2 \propto N$. For a photocurrent I_p, the variance is

$$\delta I_p^2 = 2eI_p\Delta\nu. \tag{1.8}$$

1.3 Detecting photons in the shot-noise limit

Ideally, a detector system should be limited by the statistics of the signal, say the variation in light intensity from the shot effect, and not by a thermal noise source. Let us explore this using the Current-to-Voltage Op-Amp circuit with a photodiode as the source (Figure 2), the addition of fluctuating input current for the shot noise, and a second, independent fluctuating thermal noise source from the feedback resistance (Figure 3). This pulls a lot of material together and allows us to calculate an essential quantity for any measurement, *i.e.*, the signal-to-noise ratio (S/N). For the detector, the magnitude of the output voltage from the signal I_p is

$$V_{signal} = |-R_f I_p| = R_f I_p. \tag{1.9}$$

The RMS of the fluctuating output from the two noise currents is

$$\delta V_{noise} = R_f\sqrt{\delta I_p^2 + \delta I_J^2} = R_f\sqrt{2eI_p\Delta\nu + \frac{4k_B T\Delta\nu}{R_f}} = \sqrt{2eI_p R_f\Delta\nu\left(1 + \frac{2k_B T/e}{I_p R_f}\right)} \tag{1.10}$$

where we recall that variances add in quadrature. The signal-to-noise ratio is

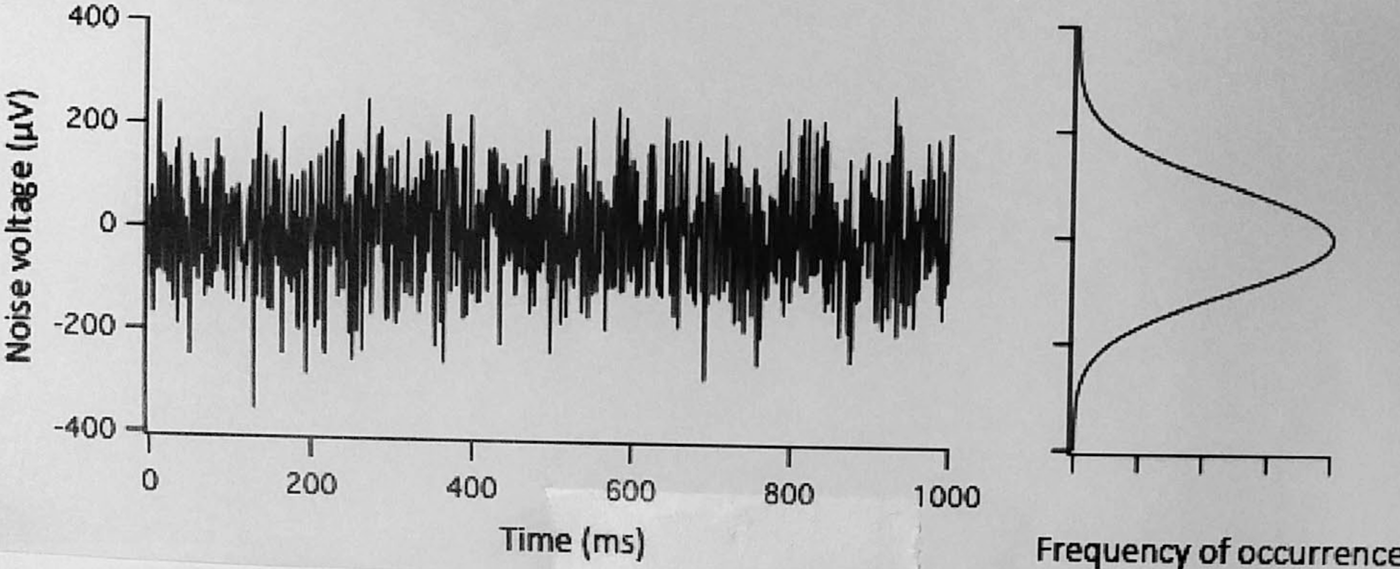
$$\frac{S}{N} \equiv \frac{V_{signal}}{\delta V_{noise}} = \frac{R_f I_p}{R_f \sqrt{2eI_p\Delta\nu\left(1+\frac{2k_BT/e}{I_pR_f}\right)}} = \sqrt{\frac{I_p}{2e\Delta\nu\left(1+\frac{2k_BT/e}{I_pR_f}\right)}} \qquad (1.11)$$

and is plotted in Figure 4. If we want to make the result independent of the thermal noise from the resistor, we must enforce the inequality $V_{signal} = I_pR_f >> 2k_BT/e$. The latter quantity is just 50 mV at room temperature. This defines a voltage scale based on fundamentals. It requires that we use a bright enough light or a large enough feedback resistor, R_f, recalling of course that any capacitance in the feedback loop, C_f, will limit the bandwidth. In this limit the signal-to-noise ratio becomes

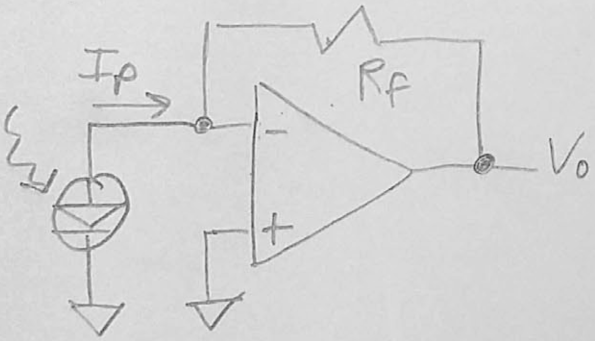
$$\frac{S}{N} \xrightarrow[I_pR_f \gg 2k_BT/e]{} \sqrt{\frac{I_p}{2e\Delta\nu}} \xrightarrow[\text{maximum bandwidth}]{} \sqrt{2R_fI_p\frac{C_f}{e}}. \qquad (1.12)$$

Thus the S/N increases as the square root of the signal strength and, from the capacitance term, as the square root of the size of the system.

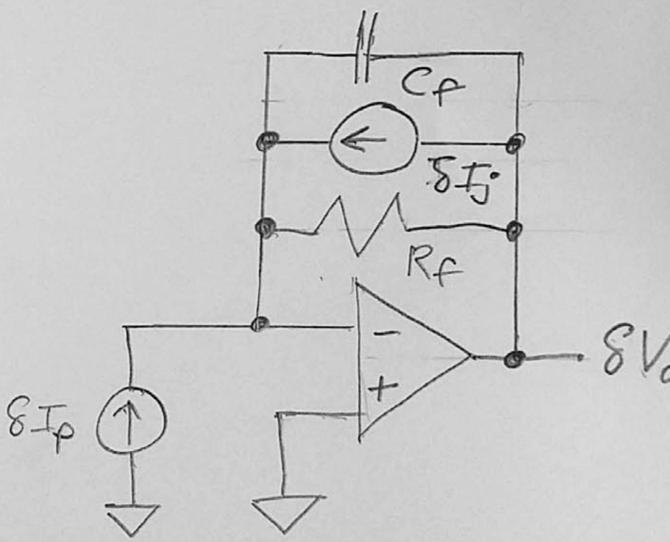
(1)



(2) Photo current-to-voltage converter using a photodiode detector



(3) Noise Model



(4)