(Corrected) script notes for Phy120 - David Kleinfeld - 03 March 2014

Note that these are "crib sheets" to aid in the presentation of maths, they are not complete lecture notes

Common Eqn (1st order linear with constant coefficient)

$$\tau \frac{dV(t)}{dt} + V(t) = F(t)$$ We let $V \equiv V(t)$ to keep sane

τ – Time constant; $F(t)$ – Drive

Homogeneous Eqn is $\frac{dV_H}{dt} + V = 0 \Rightarrow V_H(t) = \text{Const} \times e^{-t/\tau}$

Take full solution as $V(t) \equiv \alpha(t)\, V_H(t)$

$$\text{"} \quad = \alpha(t)\, e^{-t/\tau}$$

Then $\tau e^{-t/\tau} \frac{d\alpha}{dt} - \alpha e^{-t/\tau} + \alpha e^{-t/\tau} = F(t)$

Or $\frac{d\alpha}{dt} = \frac{e^{t/\tau}}{\tau} F(t) \Rightarrow \alpha(t) = \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Const}$

$$V(t) = \underbrace{e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t')}_{\int_0^t \frac{dt'}{\tau} e^{-(t-t')} F(t')} + V(t_0)\, e^{-t/\tau}$$

$V(t_0)\, e^{-t/\tau}$ – Transient

Convolution to get steady state.

1) Example of Step input, $F(t) = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}$

2) Example of sine wave input, $F(t) = \begin{cases} 0 & t < 0 \\ V_0 \sin \omega t & t \geq 0 \end{cases}$

there is only a driven response:

$$V(t) = V_0\, e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \sin \omega t'$$

$$\text{"} \quad = V_0\, e^{-t/\tau} \times \frac{1}{2i} \int_0^{t/\tau} dx\, e^{x} \left[e^{ix\omega\tau} - e^{-ix\omega\tau} \right]$$

$$V(t) = V_0 \frac{e^{-t/\tau}}{2i}\left[\int_0^{t/\tau} dx\, e^{(1+i\omega\tau)x} - \int_0^{t/\tau} dx\, e^{(1-i\omega\tau)x}\right]$$

$$" = V_0 \frac{e^{-t/\tau}}{2i}\left[\frac{e^{t/\tau}e^{i\omega t}-1}{1+i\omega\tau} - \frac{e^{t/\tau}e^{-i\omega t}-1}{1-i\omega\tau}\right]$$

$$" = V_0 \frac{1}{2i}\left[\left(\frac{e^{i\omega t}}{1+i\omega\tau} - \frac{e^{-i\omega t}}{1-i\omega\tau}\right) - \left(\frac{e^{-t/\tau}}{1+i\omega\tau} - \frac{e^{-t/\tau}}{1-i\omega\tau}\right)\right]$$

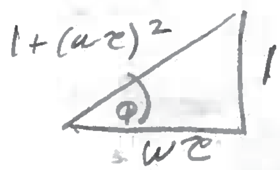
$$" = V_0 \frac{1}{2i}\left[\frac{(1-i\omega\tau)e^{i\omega t} - (1+i\omega\tau)e^{-i\omega t} + 2i\omega\tau e^{-t/\tau}}{1+(\omega\tau)^2}\right]$$

$$" = \frac{V_0}{1+(\omega\tau)^2}\left[\frac{e^{i\omega t}-e^{-i\omega t}}{2i} + \omega\tau\frac{e^{i\omega t}-e^{-i\omega t}}{2} + \omega\tau e^{-t/\tau}\right]$$

transient

$$V_{ss}(t) = V_0\left\{\left[\frac{1}{1+(\omega\tau)^2}\right]\sin\omega t + \left[\frac{\omega\tau}{1+(\omega\tau)^2}\right]\cos\omega t\right\}$$

steady state part of response



Recall $\sin(\omega t+\phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$

$\therefore \tan\phi = \omega\tau$ or $\phi = \tan^{-1}(\omega\tau)$

$$\text{Magnitude} = \sqrt{\frac{1^2+(\omega\tau)^2}{[1+(\omega\tau)^2]^2}} = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

$$\therefore V_{ss}(t) = V_0\frac{\sin[\omega t + \tan^{-1}(\omega\tau)]}{\sqrt{1+(\omega\tau)^2}}$$

$$" \xrightarrow[\omega\to 0]{} V_0 \sin\omega t$$

$$" \xrightarrow[\omega\to\infty]{} \frac{V_0\cos\omega t}{\omega\tau}$$