Problem 1 (KG).

We add impedances (Z) in the same way we do for resistors. Below are the following impedances for resistors, capacitors, and inductors respectively:

$$Z_R = R \qquad Z_C = \frac{1}{i\omega C} \qquad Z_L = i\omega L$$

Going from right to left on the circuit given, we see that we have a capacitor and resistor in parallel ($Z_R \parallel Z_C$). The impendance here is

$$Z_R \parallel Z_C : \qquad \left(\frac{1}{Z_R} + \frac{1}{Z_C}\right)^{-1} = \left(\frac{1}{R} + i\omega C\right)^{-1} = \frac{R}{1 + i\omega RC}$$

We have another resistor in series with $Z_R \parallel Z_C$. This means that the total complex impedance is:

$$\begin{aligned} Z(\omega) &= Z_R + (Z_R \parallel Z_C) = R\left(1 + \frac{1}{1 + i\omega RC}\right) \\ &= \boxed{\frac{2R + i\omega R^2 C}{1 + i\omega RC}} \end{aligned}$$

Problem 2 (KG/SH).

Recall that for a complex number $u = x + iy$ and its complex conjugate $u^* = x - iy$, the magnitude of u is $|u| = \sqrt{uu^*} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2 + y^2}$. To find $|Z(\omega)|$, we first rewrite $Z(\omega)$ to clearly see its imaginary and real parts:

$$\begin{aligned} Z(\omega) &= R\left(1 + \frac{1}{1 + i\omega\tau}\right) && \text{(substitution: } \tau = RC\text{)} \\ &= R\left[1 + \left(\frac{1}{1 + i\omega\tau}\right)\left(\frac{1 - i\omega\tau}{1 - i\omega\tau}\right)\right] && \text{(multiply by '1')} \\ &= R\left(\frac{2 + \omega^2\tau^2 - i\omega\tau}{1 + \omega^2\tau^2}\right) \\ \implies |Z(\omega)| &= \boxed{\frac{R\sqrt{(2 + \omega^2\tau^2)^2 + (\omega\tau)^2}}{1 + \omega^2\tau^2}} && \text{(magnitude)} \end{aligned}$$

Alternatively, directly evaluating from $|u| = \sqrt{uu^*}$:

$$\begin{aligned} |Z(\omega)| &= R\sqrt{\left(\frac{2 + i\omega\tau}{1 + i\omega\tau}\right)\left(\frac{2 - i\omega\tau}{1 - i\omega\tau}\right)} \\ &= \boxed{R\sqrt{\frac{4 + \omega^2\tau^2}{1 + \omega^2\tau^2}}} && \text{(magnitude)} \end{aligned}$$

These two statements are completely equivalent.

In this class, we generally have defined the break frequency equivalently to the cutoff frequency. The cutoff frequency is when the transfer function begins to attenuate, and we tend to look for the $f_{3\text{dB}}$ point because this is when the power has been reduced by 1/2. However, the true definition of a break frequency is slightly different and generally refers to just the frequencies at which the Bode Plot of the magnitude begins to change slope. Evaluating the plot below, we see there are two such break frequencies.
Solving analytically for $|Z(\omega)| = 1/\sqrt{2}$ gives imaginary frequencies for any values $R > 1$. The fact that this method doesn't work makes sense though, as we should approach this problem as we did for the LCR circuit in Lab 2 and compare $1/\sqrt{2}$ relative to the DC value, which is $2R$ in this case. Solving for $|Z(\omega)| = 2R/\sqrt{2}$ instead yields $\omega = \sqrt{2}/RC$. However, visually, this does not really make much sense, as it ends up being right in the center of the drop. Additionally, this doesn't really have a physical meaning, as the reason we look for the 3dB drop off point is because of power considerations in true transfer functions, but this problem simply asks about the impedance.

We now make a bode plot using $R = 1k\Omega$ and $C = 0.01\mu F$. (Any values for R and C that you've used in your lab are fine as long as you achieve this general shape for your curve.) Visually inspecting this Bode Plot, we can pick points to set as the break frequencies. Using the complex poles and zeroes as my reference points, I choose $\omega_1 = 1/RC$ and $\omega_2 = 2/RC$. Any two reasonably chosen break frequencies is fine so long as it has been justified properly.

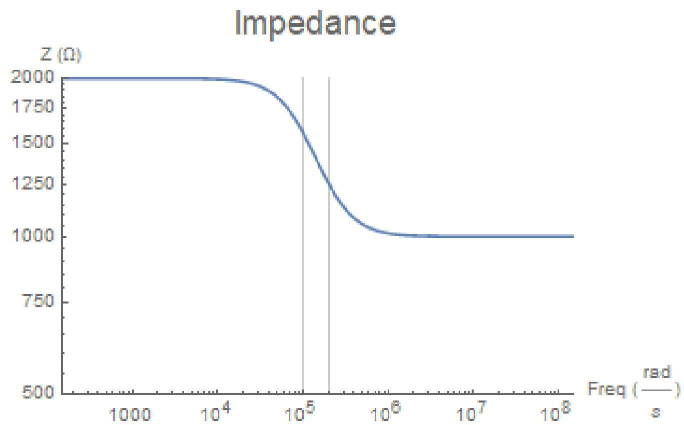


Figure 1: Bode plot for $|Z(\omega)|$. Between the two break freqs, ω_1 and ω_2, the plot has a downwards slope.

We now consider two limiting cases:

Case 1 $\omega \to 0$

$$\lim_{\omega \to 0} |Z(\omega)| = 2R$$

This is because all ω terms vanish. This is also exactly what we see above in our bode plot since $2R = 2000\Omega$.

Case 2 $\omega \to \infty$

$$\lim_{\omega \to \infty} |Z(\omega)| = R$$

In the numerator under the radical, the ω^4 term will dominate all other terms. Because the ω^2 term in the denominator will dominate, these two terms will cancel, leaving only a 1. In the plot, we see the magnitude approach $R = 1000\Omega$, as expected.

Alternatively, consider how passive elements individually behave at DC and large frequencies. Resistors are frequency dependent by capacitors act like open circuits at DC and short circuits at large ω. Thus, at DC, the capacitor opens and we have 2 resistors in series. On the other hand, for large ω, the capacitor shorts the resistor and we effectively have only one resistor.

Problem 3 (KG/SH).

The parallel combination $Z_R \parallel Z_L$ is in series with Z_C. This means the total impedance is:

$$\begin{aligned}
Z(\omega) &= Z_C + (Z_R \parallel Z_L) \\
&= \frac{1}{i\omega C} + \left(\frac{1}{R} + \frac{1}{i\omega L}\right)^{-1} \\
&= \boxed{\frac{1}{i\omega C} + \frac{i\omega LR}{R + i\omega L}} && \text{(total impedance)} \\
&= \frac{R + i\omega L - \omega^2 LRC}{-\omega^2 LC + i\omega RC} \\
&= \left(\frac{R + i\omega L - \omega^2 LRC}{-\omega^2 LC + i\omega RC}\right)\left(\frac{-\omega^2 LC - i\omega RC}{-\omega^2 LC - i\omega RC}\right) && \text{(multiply by '1')} \\
&= \frac{-i\omega^3 L^2 C + \omega^4 L^2 RC^2 - i\omega R^2 C + i\omega^3 LR^2 C^2}{\omega^4 L^2 C^2 + \omega^2 R^2 C^2} \\
\implies |Z(\omega)| &= \boxed{\frac{\sqrt{(\omega^4 L^2 RC^2)^2 + (-\omega^3 L^2 C - \omega R^2 C + \omega^3 LR^2 C^2)^2}}{\omega^4 L^2 C^2 + \omega^2 R^2 C^2}} && \text{(magnitude)} \\
&= \boxed{\sqrt{\frac{(R - \omega^2 LRC)^2 + \omega^2 L^2}{\omega^4 L^2 C^2 + \omega^2 R^2 C^2}}} && \text{(magnitude)}
\end{aligned}$$

We also calculate the phase shift:

$$\begin{aligned}
\phi(\omega) &\equiv \arctan\left[\frac{\mathrm{Im}(Z(\omega))}{\mathrm{Re}(Z(\omega))}\right] \\
&= \boxed{\arctan\left(\frac{\omega^2 L\left(R^2 C - L\right) - R^2}{\omega^3 L^2 RC}\right)}
\end{aligned}$$

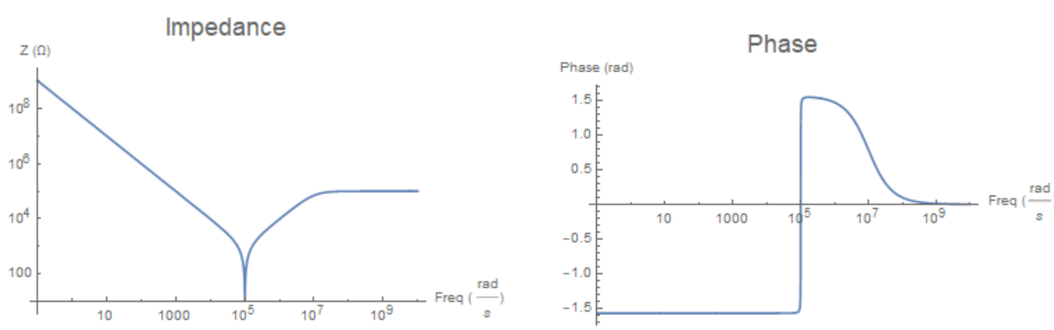


Figure 2: Bode plot for $|Z(\omega)|$. Values chosen here were $R = 100$ kΩ, $L = 10$ mH and $C = 0.01$ μF.

If we briefly consider a limiting case of $\omega \to \infty$, we know the capacitor shorts and the inductor opens. Thus, we have only an effective impedance of the resistor. We see in our magnitude plot exactly this. A resistor would cause no phase shift, as we see in the phase plot as well. Values chosen for these three components will change the specific shape of the graph slightly, but the general idea will remain the same. Conversely, at DC, the inductor shorts across the resistor, leaving only the capacitor. Capacitors act like open circuits at DC, so we expect to see our Bode Plot rising towards infinity as we go towards smaller ω, which our plot confirms.

Problem 4 (BW/SH).

When considering capacitors and inductors, we have time dependent elements which obviously do not obey Ohm's Law. Instead, we generally get differential equations. However, we can take these diff eq's to Fourier domain and find that once more, Ohm's Law holds - now as $V = IZ$ where Z is the complex impedance representation. Regardless of which method we use, we first need to solve for the transfer function of the presented LR circuit. This is simply a voltage divider using complex impedance:

In Fourier Domain: $V_{\text{out}}(\omega) = \frac{R}{R+i\omega L}V_{\text{in}}(\omega) = \frac{1}{1+i\omega\tau}V_{\text{in}}(\omega)$

$V_{\text{in}}(\omega) = \int V_o\cos(\omega_o t)e^{-i\omega t}dt = \frac{V_o}{2}\int\left(e^{-i(\omega-\omega_o)t}+e^{-i(\omega+\omega_o)t}\right)dt = V_o\pi\left(\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right)$

Inverse Transform: $V_{\text{out}}(t) = \frac{V_o\pi}{2\pi}\int\frac{e^{i\omega t}}{1+i\omega\tau}\left(\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right)d\omega = \frac{V_o}{2}\left(\frac{e^{i\omega_o t}}{1+i\omega_o\tau}+\frac{e^{-i\omega_o t}}{1-i\omega_o\tau}\right)$

Simplifying:
$$V_{\text{out}}(t) = \frac{V_o}{2}\left(\frac{(1-i\omega_o\tau)}{1+\omega_o^2\tau^2}e^{i\omega_o t}+\frac{(1+i\omega_o\tau)}{1+\omega_o^2\tau^2}e^{-i\omega_o t}\right)$$
$$= \frac{V_o}{1+\omega_o^2\tau^2}\left(\frac{(e^{i\omega_o t}+e^{-i\omega_o t})-i\omega_o\tau(e^{i\omega_o t}-e^{-i\omega_o t})}{2}\right)$$
$$= \frac{V_o}{1+\omega_o^2\tau^2}\left[\cos(\omega_o t)+\omega_o\tau\sin(\omega_o t)\right]$$

Now let's arrive at the same answer by taking the convolution integral:

$$V_{\text{out}}(t) = \int_{-\infty}^{t} dx\,(\Phi(x-t)f(t))$$

The response function, $\Phi(t)$ can be derived following equations $1.1-1.8$ on Prof. Kleinfeld's "Notes on time domain circuit analysis". This response is true in general for first order exponential responses, which is what we expect from this simple RL circuit from real world intuition and knowing that $V = L\frac{dI}{dt}$ over an inductor.

$\Phi(t) = e^{-\frac{t}{\tau}}/\tau$ Impulse Function

$\Phi(t-x) = e^{-\frac{t-x}{\tau}}/\tau$ Impulse Function Run Backwards

$f(t) = V_o\cos\omega_0 t$ Input Function

To match the solution from the Fourier Transform, we assume the input starts well in the past at $t = -\infty$. We integrate to $x = t$ as this is the point in time we are currently at coming from the past. Also put both the Response Function and Input Function in terms of a dummy variable.

$$\int_{-\infty}^{t} dx e^{-\frac{t-x}{\tau}}\frac{1}{\tau}V_o\cos\omega_0 x$$

Expand $\cos\omega_0 x$ into its exponential form:

$$\int_{-\infty}^{t} dx V_0 e^{-\frac{t-x}{\tau}}\frac{1}{\tau}\frac{e^{i\omega_0 x}+e^{-i\omega_0 x}}{2}$$

Take all of the 't' and constant terms out of the integral and do some algebra:

$$V_0\frac{e^{\frac{-t}{\tau}}}{2\tau}\int_{-\infty}^{t} dx e^{\frac{x}{\tau}}[e^{i\omega_0 x}+e^{-i\omega_0 x}]$$

The integral at this point is trivial with respect to x. Arrive at the step below:

$$V_0 \frac{e^{\frac{-t}{\tau}}}{2\tau}\left[\frac{e^{\frac{t}{\tau}} e^{i\omega_0 t}}{\frac{1}{\tau}+i\omega_0}+\frac{e^{\frac{t}{\tau}} e^{-i\omega_0 t}}{\frac{1}{\tau}-i\omega_0}\right]$$

The exponential term outside the brackets cancels with an exponent term from each of the two fractions. Expand τ into the expression inside the brackets as well:

$$V_0 \frac{1}{2}\left[\frac{e^{i\omega_0 t}}{1+i\omega_0 t}+\frac{e^{-i\omega_0 t}}{1-i\omega_0 t}\right]$$

Sum the two fractions by scaling them to a common denominator. Expand the expression and group like terms together:

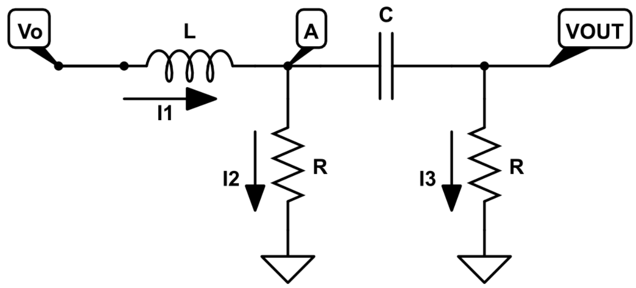
$$V_0 \frac{1}{2}\left[\frac{e^{i\omega_0 t}(1-i\omega_0\tau)+e^{-i\omega_0 t}(1+i\omega_0\tau)}{1+\omega_0^2\tau^2}\right]$$
$$V_0 \frac{1}{2}\left[\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{1+\omega_0^2\tau^2}-\frac{i\omega_0\tau(e^{i\omega_0 t}-e^{-i\omega_0 t})}{1+\omega_0^2\tau^2}\right]$$

After expanding the $\frac{1}{2}$ term, one finds that the left term is $\cos i\omega_0$ scaled by $\frac{1}{1+\omega_0^2\tau^2}$and the right term is $\sin i\omega_0$ scaled by $\frac{\omega_0\tau}{1+\omega_0^2\tau^2}$

$$V_{\text{out}}(t)=\frac{V_o}{1+\omega_o^2\tau^2}\left[\cos\left(\omega_0 t\right)+\omega_0\tau\sin\left(\omega_o t\right)\right]$$

This expression is exactly the same as what we arrived at earlier in this problem.

Problem 5 (SH).



Analyzing Kirchoff's Current Law at Node A:

$$I_1 = \frac{V_o - V_A}{Z_L} \qquad I_2 = \frac{V_A - 0}{R} \qquad I_3 = \frac{V_A - 0}{R + Z_C}$$

KCL: $I_1 = I_2 + I_3 \longrightarrow \dfrac{V_o - V_A}{Z_L} = \dfrac{V_A}{R} + \dfrac{V_A}{R + Z_C}$

Rearranging: $V_A = \dfrac{V_o}{Z_L}\left(\dfrac{1}{Z_L} + \dfrac{1}{R} + \dfrac{1}{R + Z_C}\right)^{-1}$

Output is a voltage divider: $V_{\text{out}} = \dfrac{R}{R + Z_C} V_A = \dfrac{R}{R + Z_C}\dfrac{V_o}{Z_L}\left(\dfrac{1}{Z_L} + \dfrac{1}{R} + \dfrac{1}{R + Z_C}\right)^{-1}$

Simplifying: $V_{\text{out}} = \dfrac{\frac{R}{Z_L}}{(R + Z_C)\left(\frac{1}{Z_L} + \frac{1}{R} + \frac{1}{R+Z_C}\right)} V_o = \dfrac{\frac{R}{i\omega L}}{2 + \frac{R}{i\omega L} + \frac{1}{i\omega RC} - \frac{1}{\omega^2 LC}} V_o$

Substituting $a = R/L$ and $b = 1/RC$: $V_{\text{out}} = \dfrac{\frac{a}{i\omega}}{2 + \frac{a+b}{i\omega} - \frac{ab}{\omega^2}} V_o = \dfrac{a\omega}{2i\omega^2 + (a+b)\omega - iab} V_o$

$$\therefore \boxed{V_{\text{out}}(\omega) = \frac{a\omega}{(a+b)\omega + i(2\omega^2 - ab)} V_o(\omega)}$$

Problem 6 (SH).

$$\left|\frac{V_{\text{out}}}{V_o}\right| = \sqrt{\frac{V_{\text{out}}}{V_o}\left(\frac{V_{\text{out}}}{V_o}\right)^*} = \sqrt{\frac{a^2\omega^2}{(a^2 + 2ab + b^2)\omega^2 + 4\omega^4 - 4\omega^2 ab + a^2b^2}} = \frac{a\omega}{\sqrt{4\omega^4 + (a-b)^2\omega^2 + a^2b^2}}$$

For $a = b = 1$: $\left|\dfrac{V_{\text{out}}}{V_o}\right| = \dfrac{\omega}{\sqrt{4\omega^4 + 1}}$

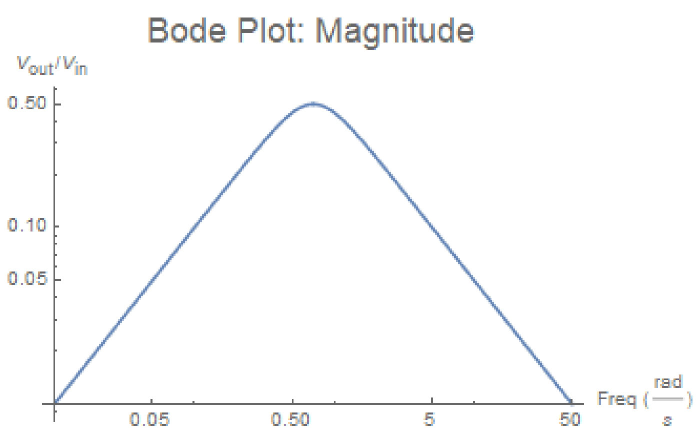


Figure 3: Bode plot for $|V_{\text{out}}(\omega)/V_o(\omega)|$. The attenuation increases before peaking and decreasing.