Problem 1 (SH).

In frequency domain, the output voltage is simply a voltage divider of the input:

$$\hat{V}_{\text{out}} = \frac{R}{R + \frac{i\omega L}{1-\omega^2 LC}} \hat{V}_{\text{in}} = \frac{R - \omega^2 RLC}{R - \omega^2 RLC + i\omega L} \hat{V}_{\text{in}}$$

$$\therefore \boxed{H(\omega) = \frac{\hat{V}_{\text{out}}}{\hat{V}_{\text{in}}} = \frac{R - \omega^2 RLC}{R + i\omega L - \omega^2 RLC}}$$

$$\left|\frac{\hat{V}_{\text{out}}}{\hat{V}_{\text{in}}}\right| = \sqrt{\frac{R - \omega^2 RLC}{R + i\omega L - \omega^2 RLC} \frac{R - \omega^2 RLC}{R - i\omega L - \omega^2 RLC}} = \boxed{\frac{|R - \omega^2 RLC|}{\sqrt{(R - \omega^2 RLC)^2 + (\omega L)^2}}}$$

$$\angle H(\omega) = \arctan\left(\frac{\text{Im}(H(\omega))}{\text{Re}(H(\omega))}\right) = \boxed{\arctan\left(\frac{\omega L}{\omega^2 RLC - R}\right)}$$

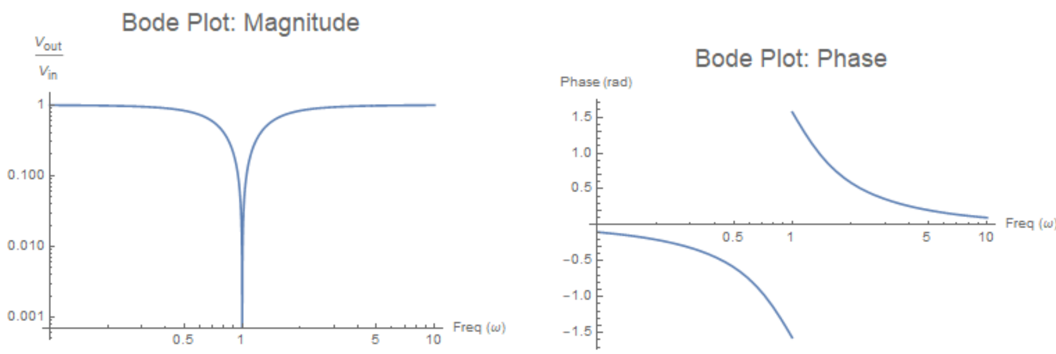


Figure 1: Bode plot for the band-stop filter. All values set to 1.

If we are given an input voltage $V_{\text{in}}(t) = V_o \sin(\omega_o t)$, we can find the resulting output using Fourier analysis, knowing the transfer function:

In Fourier Domain: $\hat{V}_{\text{out}} = \dfrac{R - \omega^2 RLC}{R - \omega^2 RLC + i\omega L} \hat{V}_{\text{in}}$

$$V_{\text{in}}(\omega) = \int V_o \sin(\omega_o t) e^{-i\omega t} dt = \frac{V_o}{2i} \int \left(e^{-i(\omega-\omega_o)t} - e^{-i(\omega+\omega_o)t}\right) dt = -V_o i\pi \left(\delta(\omega - \omega_o) - \delta(\omega + \omega_o)\right)$$

Inverse Transform:
$$V_{\text{out}}(t) = \frac{V_o \pi}{2i\pi} \int \frac{R - \omega^2 RLC}{R - \omega^2 RLC + i\omega L} e^{i\omega t} \left(\delta(\omega - \omega_o) - \delta(\omega + \omega_o)\right) d\omega$$
$$= \frac{V_o(R - \omega_0^2 RLC)}{2i} \left(\frac{e^{i\omega_o t}}{R - \omega_o^2 RLC + i\omega_o L} - \frac{e^{-i\omega_o t}}{R - \omega_o^2 RLC - i\omega_o L}\right)$$

Simplifying:
$$V_{\text{out}}(t) = \frac{V_o}{2i} \frac{R - \omega_o^2 RLC}{(R - \omega_o^2 RLC)^2 + \omega_o^2 L^2} \left[(R - \omega_o^2 RLC - i\omega_o L) e^{i\omega_o t} - (R - \omega_o^2 RLC + i\omega_o L) e^{-i\omega_o t}\right]$$
$$= \frac{R - \omega_o^2 RLC}{(R - \omega_o^2 RLC)^2 + \omega_o^2 L^2} V_o \left(\frac{(R - \omega_o^2 RLC)(e^{i\omega_o t} - e^{-i\omega_o t}) - i\omega_o L(e^{i\omega_o t} + e^{-i\omega_o t})}{2i}\right)$$
$$= \boxed{\frac{(R - \omega_o^2 RLC) V_o}{(R - \omega_o^2 RLC)^2 + \omega_o^2 L^2} \left[(R - \omega_o^2 RLC) \sin(\omega_o t) - \omega_o L \cos(\omega_o t)\right]}$$

Problem 2 (SH/BW).

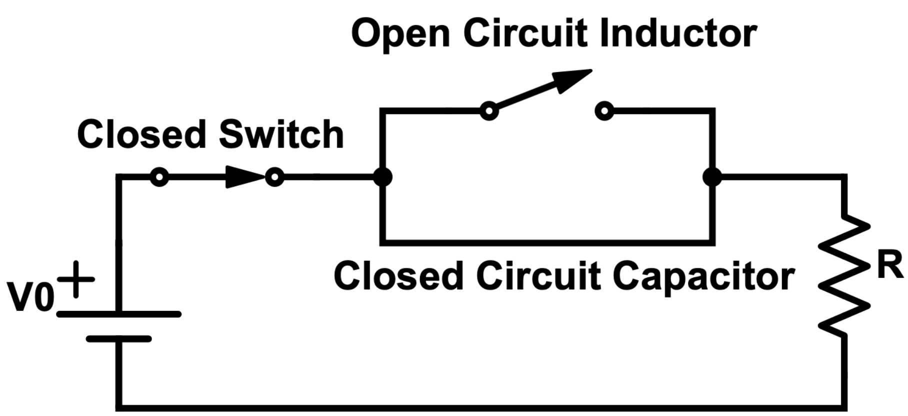
The circuit at $t = 0^+$, i.e. just after the switch is closed is portrayed below.

A capacitor functions as a short circuit wire as voltage is stepped on from 0V. A capacitor passes current relatively easily when uncharged (consistent with the stated initial condition). However, as the capacitor is charged with passing time, it becomes increasingly harder to pass current through it; it eventually functions as an open circuit.

An inductor functions as an open circuit as the switch is closed because the inductor opposes immediate current change. A magnetic field per Lenz's Law conspires to resist step increases in current through the coils. Eventually, the inductor acclimates to the magnitude of current and functions as a wire.

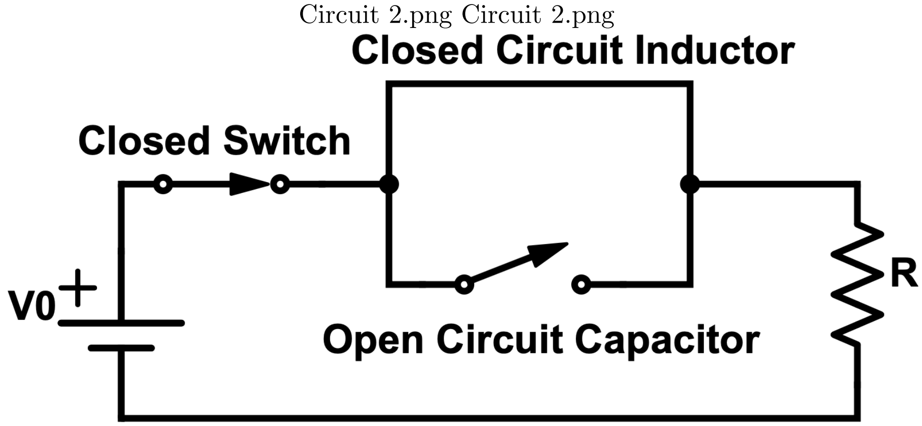
Please keep in mind that supplemental information (particularly regarding the calculus) can be found within the Laplace Transform notes posted on the PHYS 120 website.

Circuit 1.png Circuit 1.png



The circuit at $t = \infty$ is the following:

Circuit 2.png Circuit 2.png



The circuit equation describes the current running through the circuit as a function of time. From this current, you can figure out voltages dropped over certain components such as the voltage over the resistor that you solved for in the previous problem. Keep in mind that the switch closes at $t = 0$.

From the concepts elaborated in the "Notes on Laplace circuit analysis" document on the PHYS 120 website, one can transform impedances, currents, and voltages into Laplace-space to simplify computation. Taking the Inverse Laplace Transform brings us back into the time-domain to arrive at our final solution.

$$\begin{aligned} V(s) &= \int_{-\infty}^{\infty} dtv(t)e^{-a}e^{-i\omega t} \\ &= \int_{0}^{\infty} dtv(t)e^{st} \end{aligned}$$

Let's solve for current in Laplace-space, $I(s)$. The final answer is the voltage over the resistor, $V(s)$, which is equivalent to $I(s)R$.

The switch closes at $t = 0$ so we can set the lower limit of integration to 0. This implies that we account for initial conditions, which are stated in the problem: $V_c(0) = 0$

Let's determine impedances of the inductor and capacitor in Laplace-space (referred to as s-space henceforth):
The inductor voltage follows the time dependant relation, $V(t) = L\frac{dI(t)}{dt}$. Knowing that taking the time derivative of $V(s)$ in general yields the original $V(s)$ scaled by a factor of s, we infer that taking the time derivative of $I(s)$ works the same way. Thus, $V(s) = LsI(s)$ for an inductor.

The capacitor voltage follows the time dependant relation, $V(t) = \frac{1}{C}\int dtI(t)$. Following a similar argument as the inductor, we find that $V(s) = \frac{1}{C}\frac{1}{s}I(s)$.

We find that the general impedances of the inductor and capacitor in s-space are $Z(s) = Ls$ and $Z(s) = \frac{1}{Cs}$, respectively. R is time independent and thus does not transform into anything different in s-space.

L and C are in parallel, so we can find an equivalent impedance that accounts for both components:

$$\begin{aligned} Z_{eq}(s) &= \left(\frac{1}{Ls} + Cs\right)^{-1} \\ &= \frac{Ls}{1 + CLs^2} \end{aligned}$$

We can now write down a voltage relation in s-space consisting of a KVL loop around the entire circuit:

$$0 = \frac{-V_0}{s} + \frac{sL}{1 + CLs^2}I(s) + I(s)R$$

Isolate $I(s)$ to get

$$\begin{aligned} I(s) &= \frac{V_0}{s}\left(\frac{sL}{s^2CL + 1} + R\right)^{-1} \\ &= V_0\left(\frac{s^2CL + 1}{s(R + sL + CLRs^2)}\right) \end{aligned}$$

$$\boxed{V(s) = RV_0\left(\frac{s^2CL + 1}{s(R + sL + CLRs^2)}\right)}$$

To solve for $V(t)$, we need to take the Inverse Laplace Transform, which is given by a contour integral in the complex s-plane:

$$V(t) = \frac{1}{2\pi i}\int_C dsV(s)e^{st}$$

The easiest way to solve one of these integrals is to set up $V(s)$ as a fraction so that the denominator is a product of complex zeroes; we need certain values of s (poles) to have the expression blow up to infinity. This way, we can use the Cauchy Residue theorem to pretty much skip all of the actual calculus.
To simplify algebra, let's adapt V(s) from the intermediate expression derived a few steps prior:

$$\begin{aligned} V(s) &= \frac{RV_0}{s}\left(\frac{sL}{s^2CL+1}+R\right)^{-1} \\ &= V_0\left(\frac{1}{s}-\frac{L}{RCLs^2+Ls+R}\right) \\ &= V_0\left(\frac{1}{s}-\frac{L}{RCL(s^2+\frac{s}{CL}+\frac{1}{CL})}\right) \end{aligned}$$

At this point, we can invoke two Inverse Laplace Transform operations on the two fractions separately. The first fraction is trivial and the Inverse Transform returns V_0. The second one will require the Cauchy Residue Theorem.

Use the quadratic formula to find the roots, a and a^*, of $s^2+\frac{s}{RC}+\frac{1}{CL}$. Put it in the form $A+Bi$.

$$\begin{aligned} a &= \frac{\frac{1}{-CL}+i\sqrt{-\frac{1}{R^2C^2}+\frac{4}{CL}}}{2} \\ a^* &= \frac{\frac{1}{-CL}-i\sqrt{-\frac{1}{R^2C^2}+\frac{4}{CL}}}{2} \end{aligned}$$

$$\begin{aligned} V(t) &= V_0-\frac{1}{2\pi i}\int ds\frac{V_0}{RC}\left(\frac{1}{(s-a)(s-a^*)}\right)e^{st} \\ &= V_0-\frac{1}{2\pi i}\frac{V_0}{RC}(2\pi i)\left[\frac{e^{at}}{s-a}+\frac{e^{a^*t}}{s-a^*}\right] \\ &= V_0-\left(\frac{e^{\left(\frac{1}{2}t\left(-\frac{1}{CR}-i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)\right)}}{CR\left(\frac{1}{2}\left(\frac{1}{CR}-i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)+\frac{1}{2}\left(-\frac{1}{CR}-i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)\right)}\right) \\ &- \left(\frac{e^{\left(\frac{1}{2}t\left(-\frac{1}{CR}+i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)\right)}}{CR\left(\frac{1}{2}\left(\frac{1}{CR}+i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)+\frac{1}{2}\left(-\frac{1}{CR}+i\sqrt{\frac{4}{CL}-\frac{1}{C^2R^2}}\right)\right)}\right) \end{aligned}$$

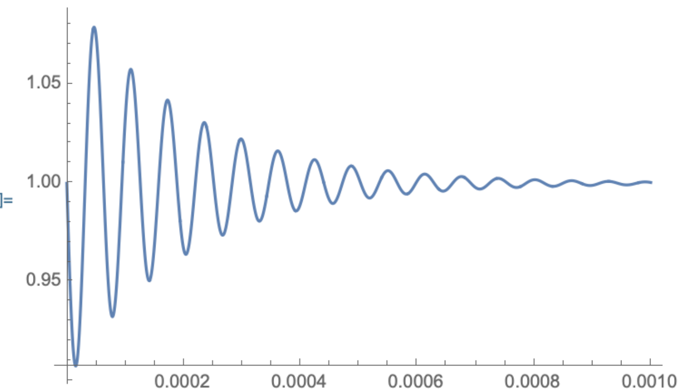
$$\boxed{=\frac{2e^{-\frac{t}{2CR}}\sin\left(\frac{t\sqrt{\frac{4C}{L}-\frac{1}{R^2}}}{2C}\right)}{R\sqrt{\frac{4C}{L}-\frac{1}{R^2}}}}$$

We can see from the below plot the time dependent voltage across the resistor as everything settles after the step voltage input (closing the switch). This is taken at $V_0 = 1V$ and at arbitrary R, C, L values such that $\frac{4C}{L} > \frac{1}{R^2}$. (underdamped). There is a noticeable exponential increase in voltage measured over the resistor until it levels out at V_0. This is consistent with the fact that reactive elements ideally possess zero real resistance.

Final Plot.png Final Plot.png

In[27]:= $$\text{Plot}\left[1-\frac{2\,e^{-\frac{t}{2cR}}\,\text{Sin}\left[\frac{\sqrt{\frac{4c}{L}-\frac{1}{R^2}}\,t}{2c}\right]}{\sqrt{\frac{4c}{L}-\frac{1}{R^2}}\,R},\ \{t,0,0.001\},\ \text{PlotRange}\to\text{All}\right]$$

Out[27]=

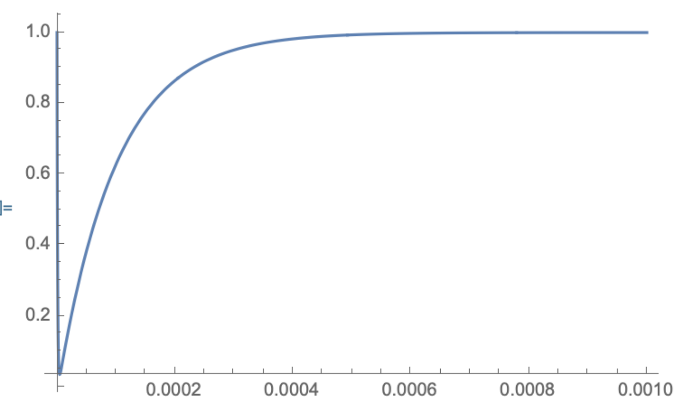


The next plot shows the same overall time dependance as the previous term. However, R, C, and L are chosen such that $\frac{4C}{L} < \frac{1}{R^2}$ (overdamped); there are no oscillations observed.

Circuit 1 Overdamped.png Circuit 1 Overdamped.png

In[12]:= $$\text{Plot}\left[1-\frac{2\,e^{-\frac{t}{2cR}}\,\text{Sin}\left[\frac{\sqrt{\frac{4c}{L}-\frac{1}{R^2}}\,t}{2c}\right]}{\sqrt{\frac{4c}{L}-\frac{1}{R^2}}\,R},\ \{t,0,0.001\},\ \text{PlotRange}\to\text{All}\right]$$

Out[12]=



Problem 3 (KG).

Recall that for a Fourier series expansion for a function $V(t)$, our complex coefficients ($\tilde{c}_k$) are defined as

$$\tilde{c}_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V(t)e^{-ik\omega_0 t}dt \quad \rightarrow \quad V(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{ik\omega_0 t}$$

We are given $\omega_0 = \frac{2\pi}{T}$ for

$$V(t) = \begin{cases} V_0 \sin(\omega_0 t) & 0 < t \leq \frac{T}{4} \\ 0 & \frac{T}{4} < t \leq T \end{cases}$$

we see that we only need to integrate $\tilde{c}_k$ on the periodic domain $0 < t \leq \frac{T}{4}$ since the other integrals will yield zero.

$$\begin{aligned}
\tilde{c}_k &= \frac{1}{T}\int_0^{\frac{T}{4}} V_0 \sin(\omega_0 t)e^{-ik\omega_0 t}dt \\
&= \frac{V_0}{T}\int_0^{\frac{T}{4}} \frac{(e^{i\omega_0 t} - e^{-i\omega_0 t})e^{-ik\omega_0 t}}{2i}dt && \left(\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}\right) \\
&= \frac{V_0}{T2i}\int_0^{\frac{T}{4}} \left(e^{i\omega_0 t(1-k)} - e^{-i\omega_0 t(1+k)}\right)dt \\
&= \frac{V_0}{T2i}\left(\frac{e^{i\omega_0 t(1-k)}}{i\omega_0(1-k)} + \frac{e^{-i\omega_0 t(1+k)}}{i\omega_0(1+k)}\right)\Bigg|_0^{\frac{T}{4}} \\
&= \frac{V_0}{T2\omega_0}\left(\frac{(1+k)e^{i\omega_0 t(1-k)} + (1-k)e^{-i\omega_0 t(1+k)}}{k^2-1}\right)\Bigg|_0^{\frac{T}{4}} && \text{(distribute (-1) in denominator)} \\
&= \frac{V_0}{2T\omega_0}\left(\frac{e^{-i\omega_0\frac{T}{4}k}[e^{i\omega_0\frac{T}{4}} + e^{-i\omega_0\frac{T}{4}} + k(e^{i\omega_0\frac{T}{4}} - e^{-i\omega_0\frac{T}{4}})] - 2}{k^2-1}\right) && \left(\cos(x) = \frac{e^{ix} + e^{-ix}}{2}\right) \\
&= \boxed{\frac{V_0}{T\omega_0}\left(\frac{e^{-i\omega_0\frac{T}{4}k}[\cos(\omega_0\frac{T}{4}) + ik\sin(\omega_0\frac{T}{4})] - 1}{k^2-1}\right) \text{ for } k \in \mathbb{Z}, k \neq \pm 1} && \text{(in terms of } T \text{ and } \omega_0\text{)} \\
&= \frac{V_0}{2\pi}\left(\frac{ike^{-ik\frac{\pi}{2}} - 1}{k^2-1}\right) && \left(\text{substitute } \omega_0 = \frac{2\pi}{T}\right) \\
&= \boxed{\frac{V_0}{2\pi}\left(\frac{(-1)^k(i^{k+1})k - 1}{k^2-1}\right) \text{ for } k \in \mathbb{Z}, k \neq \pm 1} && \left(ie^{-\frac{ik\pi}{2}} = i(e^{-\frac{i\pi}{2}})^k = i(-i)^k = (-1)^k(i^{k+1})\right)
\end{aligned}$$

Since $k = \pm 1$ gives us an infinite expression, we must find $\tilde{c}_{\pm 1}$ explicitly.

$$\tilde{c}_1 = \int_0^{\frac{T}{4}} V_0 \sin(\omega_0 T)e^{-i\omega_0 t}dt = \boxed{\frac{V_0}{8\pi}(2 - i\pi) \qquad k = 1}$$

$$\tilde{c}_{-1} = \int_0^{\frac{T}{4}} V_0 \sin(\omega_0 T)e^{i\omega_0 t}dt = \boxed{\frac{V_0}{8\pi}(2 + i\pi) \qquad k = -1}$$