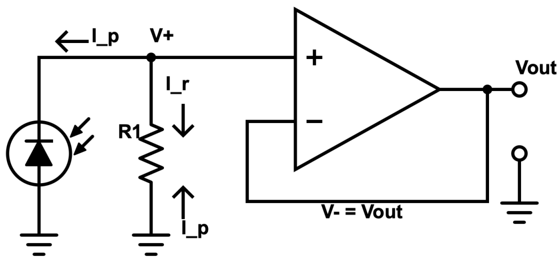

Figure 1: Top: Circuit given by the problem labelled with currents and node voltages.

Problem 1.

The circuit looks like the photodiode circuit from HW4. However, one significant difference is that V_+ is now the common terminal instead of V_-, and V_- is linked directly with V_{out} rather than grounded. This means that the amplifier is not inverting and the current flowing into V_+ from either side do not necessarily have to cancel. Recall that photodiodes operate in reverse bias, producing negative current, I_p. We arbitrarily define the direction of I_r according to physical intuition. Writing the KCL loop to find current polarity:

$$I_p + I_r = 0$$
$$I_p = -I_r$$

We see that V_+ equates to $I_r * R$ which is equivalent to saying $V_+ = -I_p * R$.

We recognize this addition of a differential op amp as non-inverting and performing an operation of unity gain. This is shown below:

$$A(V_+ - V_-) = V_{out}$$
$$V_- = V_{out}$$
$$A(V_+ - V_{out=} = V_{out}$$
$$V_{out} = \frac{A}{A+1} V_+$$

Thus, $\boxed{V_{out} = -\frac{A}{A+1} I_p * R \approx -I_p * R}$

Problem 2.

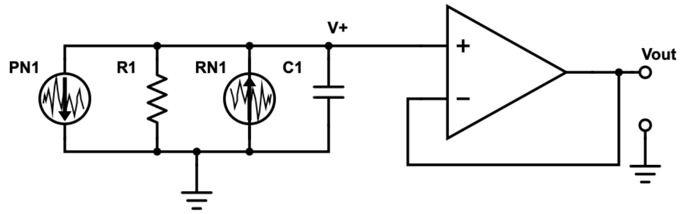

Figure 2: The noise generated from the photodiode (shot noise) and resistor (thermal noise) are modelled as ideal current sources. The resistor is also decomposed into its practical limitations as an ideal capacitor in parallel with an ideal resistor. The inductance of the resistor is neglected. The direction of thermal noise is dictated by the direction of shot noise due to the passivity of the resistor

To understand why the capacitor is in parallel with the resistor, think of how the wires/leads terminate at the ends of the resistive material as if you were probing it with a voltmeter (which was in parallel, right?). The two wire ends end ”on top” of the resistor and form an open circuit that is certainly not in series with the resistive material.

The polarity of noise current is the same as that of the majority current flowing through the circuit. Since the photodiode produces negative current, the direction of the current points opposite to the direction of the diode on the schematic, down. With KCL logic, the direction of the resistor current (divided between the resistor and the capacitor) points up.

Problem 3.

The derivation of thermal and shot noise is detailed in your ”Noise and signal-to-noise handout.” NOTE THAT THERE IS A TYPO IN THE HANDOUT WHEN SOLVING FOR COMPLEX IMPEDANCE OF THE AMPLIFIER. Solve it for yourself or scroll down to find out what it should be.

The handout derives the variance of voltage in the perspective of frequency bandwidth. This is expressed as the integral (sum) of the entire spectrum of frequencies relevant to the complex impedance of the low pass filter (R parallel with C) equivalent circuit: $\Delta\nu = \frac{1}{2\pi}\int_0^\infty d\omega \frac{1}{1+(\frac{\omega}{RC})^2} = \frac{1}{4RC}$.

It is briefly mentioned that thermodynamics can also be used to derive the expression for thermal noise. This is briefly stated below:

The first goal is to prove the statement made in the handout:

$$\frac{1}{2}CV^2 = \frac{1}{2}k_BT$$

We know from Maxwell-Boltzmann statistics that the probability of something happening at a certain energy takes the form: $e^{\frac{-Energy}{k_BT}}$. What is this Energy?
Resistors do not store energy ideally. Therefore, in steady state conditions, the equivalent capacitor stores all of the energy ”present in” the resistor. $Energy = \frac{1}{2}CV^2$. Probability is a function of voltage in this case.

$$dProbability = dP(V) = A * e^{\frac{-CV^2}{2k_BT}} dV$$

Where A is a normalization constant

Perform a change of variables from V to x to simplify the Gaussian integral (the exponential term):

$$x^2 = \frac{CV^2}{2k_BT}$$
$$2xdx = \frac{CV}{k_BT}dV$$
$$dV = \frac{2k_BT}{C}\frac{x}{V}dx$$

Plug x back into dV to get

$$dV = \sqrt{\frac{2k_BT}{C}}dx$$

Solve for the normalization constant by integrating over all voltages and setting the probability equal to 1.

$$\int_{-\infty}^{\infty} dP(V) = \int_{-\infty}^{\infty} dV A * e^{\frac{-CV^2}{2k_BT}} = 1$$
$$A * \sqrt{\frac{2k_BT}{C}} * \int_{-\infty}^{\infty} dx e^{-x^2} = 1$$
$$A * \sqrt{\frac{2k_BT}{C}} * \sqrt{\pi} = 1$$
$$A = \sqrt{\frac{C}{2\pi k_BT}}$$

The V^2 in the capacitor potential energy is actually the mean squared voltage: $\overline{V^2}$. Solving for average squared voltage here is analogous to how you would do so in Quantum Mechanics. Using the same normalization constant:

$$\overline{V^2} = A * \int_{-\infty}^{\infty} dV * V^2 * e^{\frac{-CV^2}{2k_BT}}$$
$$= \sqrt{\frac{C}{2\pi k_BT}} * \sqrt{\frac{2k_BT}{C}} * \int_{-\infty}^{\infty} dx x^2 e^{-x^2}$$
$$= \sqrt{\frac{C}{2\pi k_BT}} * \sqrt{\frac{2k_BT}{C}} * \sqrt{\frac{\pi}{2}}$$
$$= \frac{k_BT}{C}$$

As you can see from above, $C\overline{V^2} = k_BT$. Dividing both sides by 1/2 brings us to the Ansatz concerning the Equipartition Theorem that the handout assumed. If you were not familiar with the Calculus done here, you will have many more opportunities to do this in PHYS 140A and PHYS 130A/B.

Recall that no energy is stored in the resistor; only the capacitor stores energy. Now, modelling the practical resistor as an ideal resistor in parallel with an ideal capacitor like before, the voltage stored in the capacitor, V_C, comes out to be:

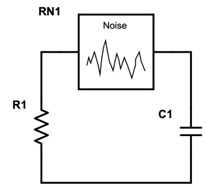

Figure 3: The Thevenin equivalent of the thermal noise source in Figure 2 can be modelled as a voltage source in series with the resistor. We are only concerned with this voltage loop

$$\begin{aligned}\overline{V_C} &= \overline{V_{Noise}} * \frac{Z_C}{Z_C + Z_R} \\ &= \overline{V_{Noise}} * \frac{1/(i\omega C)}{1/(i\omega C) + R} \\ &= \overline{V_{Noise}} * \frac{1}{1 + i\omega RC} \\ \overline{V_C^2} &= \overline{V_{Noise}^2} * \frac{1}{1 + (\omega RC)^2}\end{aligned}$$

Note that voltage stored in the capacitor is frequency dependent. Greater frequencies bring V_C closer to 0. We're getting back into the business of bandwidth, $\Delta\nu$, but from a different approach. We see that the voltage over the capacitor is limited to certain frequencies between 0 and ∞.

$\overline{V_C^2}$ includes the noise compounded over every possible frequency (similar to the concept of bandwidth). Evaluate this by taking the integral over frequency:

$$B * \overline{V_{Noise}^2} df = d\overline{V_C^2}$$

Where B is some prefactor after changing variables into f

$$\int_0^\infty \overline{V_C^2} = \overline{V_{Noise}(0)^2} \int_0^\infty \frac{df}{1 + (\omega RC)^2}$$

$$\begin{aligned}x &= \omega RC = 2\pi f RC \\ df &= \frac{dx}{2\pi RC}\end{aligned}$$

Transform the integral into x-space and equate it to the value obtained through the Equipartition Theorem:

$$\begin{aligned}\overline{V_C^2} &= \frac{\overline{V_{Noise}(0)^2}}{2\pi RC} \int_0^\infty \frac{dx}{1 + x^2} \\ &= \frac{\overline{V_{Noise}(0)^2}}{2\pi RC} * \frac{\pi}{2} \\ &= \frac{V_{Noise}(0)^2}{4RC} = \frac{k_B T}{C} \\ \overline{V_{Noise}(0)^2} &= \overline{\delta V(0)^2} = 4Rk_BT\end{aligned}$$

Notice that $\overline{\delta V(0)^2} = \delta V_J^2$ from the notes, but it is missing a factor of $\Delta\nu$. This is because $\overline{\delta V(0)^2}$ is taken at one particular frequency, $f = 0$. Accounting for all of the frequencies yields $\boxed{\delta V_{Thermal}(f)^2 = 4Rk_BT\Delta\nu}$.

Now that we've addressed thermal noise, we need to consider shot noise.
The notes state that the variance of shot noise current is $\delta I_p^2 = I_p * 2e * \Delta\nu$. This can be interpreted as the charge of each photon (2 electrons) times the bandwidth of interest. I_p and δI_p scale with photon count. These values also scale with the value of I_p and is dependent on the surrounding circuit.
What is I_p? Recall the photodiode current to voltage converter circuit from lab section 4-4. Acknowledge that I_p is exactly equal to $\frac{V_{signal}}{R}$ – R being the resistor from Problem 2 – within the scope of Op Amp limitations.

As you know from PHYS 2CL, variances add in quadrature.

$$\begin{aligned}\delta V &= \sqrt{\delta V_{Thermal}^2 + \delta V_{photon}^2} \\ &= \sqrt{(4k_BTR) + R^2 \delta I_{photon}^2} \\ &= \boxed{\sqrt{(4k_BTR) + R^2 * (2e * \Delta\nu I_p)}}\end{aligned}$$

Practically, we have much greater control over photon count than over thermal noise, so we want thermal noise to disappear from δV. This is why we choose a large R_f (10M in the lab).

Problem 4.

Signal equals the measured voltage while Noise refers to the variance of that measurement. We know from Figure 2 that V_{signal} is measured as the voltage dropped across the resistor, or $I_p * R$. Plugging in the value for variance found in Problem 3:

$$\begin{aligned}\frac{S}{N} &= \frac{I_p * R}{\sqrt{(4k_BTR) + R^2 * (2e * \Delta\nu I_p)}} \\ &= \frac{I_p * R}{R * \sqrt{\frac{4k_BT}{R} + (2e * \Delta\nu I_p)}} \\ &= \frac{I_p}{\sqrt{\frac{4k_BT}{R} + (2e * \Delta\nu I_p)}} \\ &= \boxed{\sqrt{\frac{I_p}{2e * \Delta\nu * \left(1 + \frac{2k_BT/e}{I_pR}\right)}}}\end{aligned}$$

Problem 5.

```
In[3]:= ratio[ip_] := Sqrt[ip / (2 e Δv (1 + (2 kT/e)/(ip R)))]

In[8]:= kT = 0.025;
        e = 1;
        Δv = 1 / (4 × 1*^7 × 1*^-6);
        R = 1*^7;

In[23]:= LogLogPlot[ratio[ip], {ip, 0, 1}]
```

Out[23]=

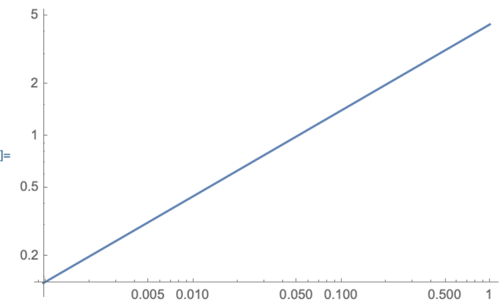

Figure 4: X Axis is the log-scaled I_p. Y-Axis is the log-scaled Signal to Noise ratio

Problems 1-3 on the handout: "Notes on an "Op-Amp Differentiator Circuit"

Problem 1 (SH).
We are given closed loop gain:

$$G(\omega) = -\frac{j\omega R_f C_{in}}{(1 + j\omega R_f C_f)(1 + j\omega R_{in} C_{in})}$$

Additionally, we also have the magnitude of the closed loop gain:

$$|G(\omega)| = \frac{\omega R_f C_{in}}{\sqrt{[1 + (\omega R_f C_f)^2]\,[1 + (\omega R_{in} C_{in})^2]}}$$

Using your favorite plotting software and choosing appropriate values, the magnitude of the Bode Plot looks like this:

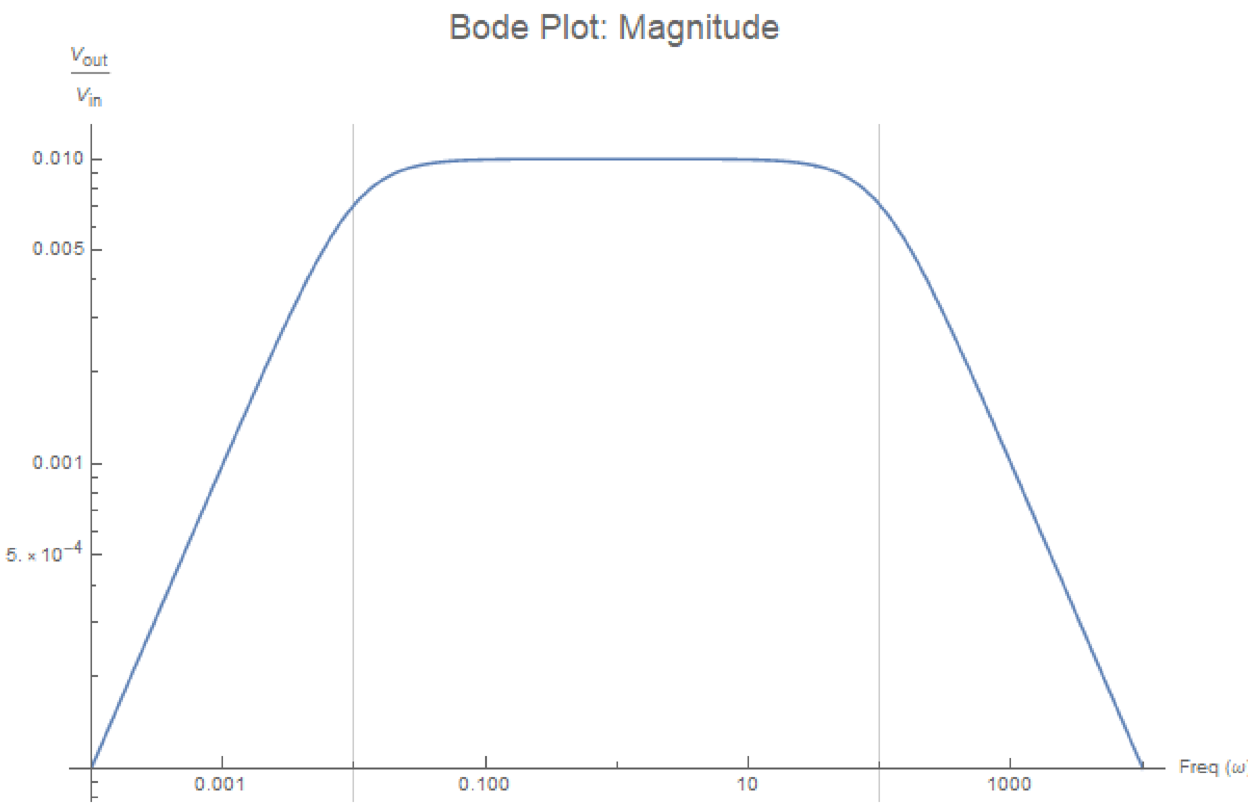

Figure 5: Bode plot. Values chosen: $R_f = 10$ kΩ and $C_f = 1$ μF, $R_{in} = 1$ MΩ and $C_{in} = 100$ μF.

We can clearly see from the transfer function that we have a zero at the origin and poles at $1/R_f C_f$ and $1/R_{in} C_{in}$. Keeping in mind $1 \gg R_{in} C_{in} \gg R_f C_f$, we expect to see the Bode Plot rising constantly until we encounter our first break frequency at $1/R_{in} C_{in}$, where the Bode Plot ought to level out. When we encounter our second break frequency at $1/R_f C_f$, we will begin to see the magnitude drop. As we can see on the plot, this is accurately represented.

For the increasing gain region, $\omega \ll \frac{1}{R_{in}C_{in}} \ll \frac{1}{R_f C_f}$, or $\omega R_f C_f \ll 1$ and $\omega R_{in} C_{in} \ll 1$, so:

$$\begin{aligned}
|G(\omega)| &= \frac{\omega R_f C_{in}}{\sqrt{[1+(\omega R_f C_f)^2]\,[1+(\omega R_{in}C_{in})^2]}} \\
&= \omega R_f C_{in}\left[1+(\omega R_f C_f)^2\right]^{-1/2}\left[1+(\omega R_{in}C_{in})^2\right]^{-1/2} && \text{Rewrite} \\
&\approx \omega R_f C_{in}\left[1-\frac{(\omega R_f C_f)^2}{2}+...\right]\left[1-\frac{(\omega R_{in}C_{in})^2}{2}+...\right] && \text{Binomial Expansion} \\
&\approx \omega R_f C_{in}\left[1-\mathcal{O}((\omega R_f C_f)^2)\right]\left[1-\mathcal{O}((\omega R_{in}C_{in})^2)\right] && \text{Ignoring higher order} \\
&\approx \omega R_f C_{in}
\end{aligned}$$

Thus, for the low frequency range, we find the slope to be $d|G(\omega)|/d\omega \approx R_f C_{in}$. On the BodePlot's log-log scaling, we would have find a relation such that $\log(y) = \log(kx) = \log(k) + \log(x)$, where x in this instance is our frequency, k is the linear slope constant, and y is the gain. Because $\log(k)$ is merely some constant, the slope of the gain on log-log scale would be $+1$.

Doing the same for the constant gain region where $\frac{1}{R_{in}C_{in}} \ll \omega \ll \frac{1}{R_f C_f}$, or $\omega R_f C_f \ll 1$ and $\omega R_{in} C_{in} \gg 1$:

$$\begin{aligned}
|G(\omega)| &= \frac{\omega R_f C_{in}}{\sqrt{[1+(\omega R_f C_f)^2]\,[1+(\omega R_{in}C_{in})^2]}} \\
&= \omega R_f C_{in}\left[1+(\omega R_f C_f)^2\right]^{-1/2}[(\omega R_{in}C_{in})^2]^{-1/2}\left[\frac{1}{(\omega R_{in}C_{in})^2}+1\right]^{-1/2} && \text{Rewrite} \\
&\approx \frac{R_f}{R_{in}}\left[1-\frac{(\omega R_f C_f)^2}{2}+...\right]\left[1-\frac{1}{2(\omega R_{in}C_{in})^2}+...\right] && \text{Binomial Expansion} \\
&\approx \frac{R_f}{R_{in}}\left[1-\mathcal{O}((\omega R_f C_f)^2)\right]\left[1-\mathcal{O}\left(\left(\frac{1}{\omega R_{in}C_{in}}\right)^2\right)\right] && \text{Ignoring higher order} \\
&\approx \frac{R_f}{R_{in}}
\end{aligned}$$

Evidently, this yields the slope for $d|G(\omega)|/d\omega \approx 0$. It also tells us the maximum gain is R_f/R_{in}.

Finally, for the decreasing gain region, $\frac{1}{R_{in}C_{in}} \ll \frac{1}{R_f C_f} \ll \omega$, or $\omega R_f C_f \gg 1$ and $\omega R_{in} C_{in} \gg 1$, so:

$$\begin{aligned}
|G(\omega)| &= \frac{\omega R_f C_{in}}{\sqrt{[1+(\omega R_f C_f)^2]\,[1+(\omega R_{in}C_{in})^2]}} \\
&= \frac{\omega R_f C_{in}}{\omega^2 R_f C_f R_{in} C_{in}}\left[\frac{1}{(\omega R_f C_f)^2}+1\right]^{-1/2}\left[\frac{1}{(\omega R_{in}C_{in})^2}+1\right]^{-1/2} && \text{Rewrite} \\
&\approx \frac{1}{\omega R_{in} C_f}\left[1-\frac{1}{2(\omega R_f C_f)^2}+...\right]\left[1-\frac{1}{2(\omega R_{in}C_{in})^2}+...\right] && \text{Binomial Expansion} \\
&\approx \frac{1}{\omega R_{in} C_f}\left[1-\mathcal{O}\left(\left(\frac{1}{\omega R_f C_f}\right)^2\right)\right]\left[1-\mathcal{O}\left(\left(\frac{1}{\omega R_{in}C_{in}}\right)^2\right)\right] && \text{Ignoring higher order} \\
&\approx \frac{1}{\omega R_{in} C_f}
\end{aligned}$$

Therefore, for the high frequency range, we find the slope to be $d|G(\omega)|/d\omega \approx -\frac{1}{R_{in}C_f\omega^2}$. On the Bode-Plot's log-log scaling, we would have find a relation such that $\log(y) = \log(1/kx) = \log(1) - \log(kx) = -\log(k) - \log(x)$, where x in this instance is our frequency, k is the linear slope constant, and y is the gain. Because $\log(k)$ is merely some constant, the slope of the gain on log-log scale would be -1.

Problem 2 (SH).
Let's rewrite our transfer function:

$$\begin{aligned}
G(\omega) &= -\frac{j\omega R_f C_{in}}{(1+j\omega R_f C_f)(1+j\omega R_{in} C_{in})} \\
&= \frac{-j\omega R_f C_{in}(1-j\omega R_f C_f)(1-j\omega R_{in} C_{in})}{(1+(\omega R_f C_f)^2)(1+(\omega R_{in} C_{in})^2)} && \text{Multiply by "1"} \\
&= \frac{-j\omega R_f C_{in}(1-j\omega(R_f C_f + R_{in} C_{in}) - \omega^2 R_f R_{in} C_f C_{in})}{(1+(\omega R_f C_f)^2)(1+(\omega R_{in} C_{in})^2)} && \text{Simplification} \\
&= \frac{\omega R_f C_{in}}{(1+(\omega R_f C_f)^2)(1+(\omega R_{in} C_{in})^2)}\left[-\omega(R_f C_f + R_{in} C_{in}) + j(\omega^2 R_f R_{in} C_f C_{in} - 1)\right]
\end{aligned}$$

From this form, we can quickly find the phase:

$$\angle G(\omega) = \arctan\left(\frac{\text{Im}[G(\omega)]}{\text{Re}[G(\omega)]}\right) = \arctan\left(\frac{1-\omega^2 R_f R_{in} C_f C_{in}}{\omega(R_f C_f + R_{in} C_{in})}\right)$$

Problem 3 (SH).

Plotting the phase, we find:

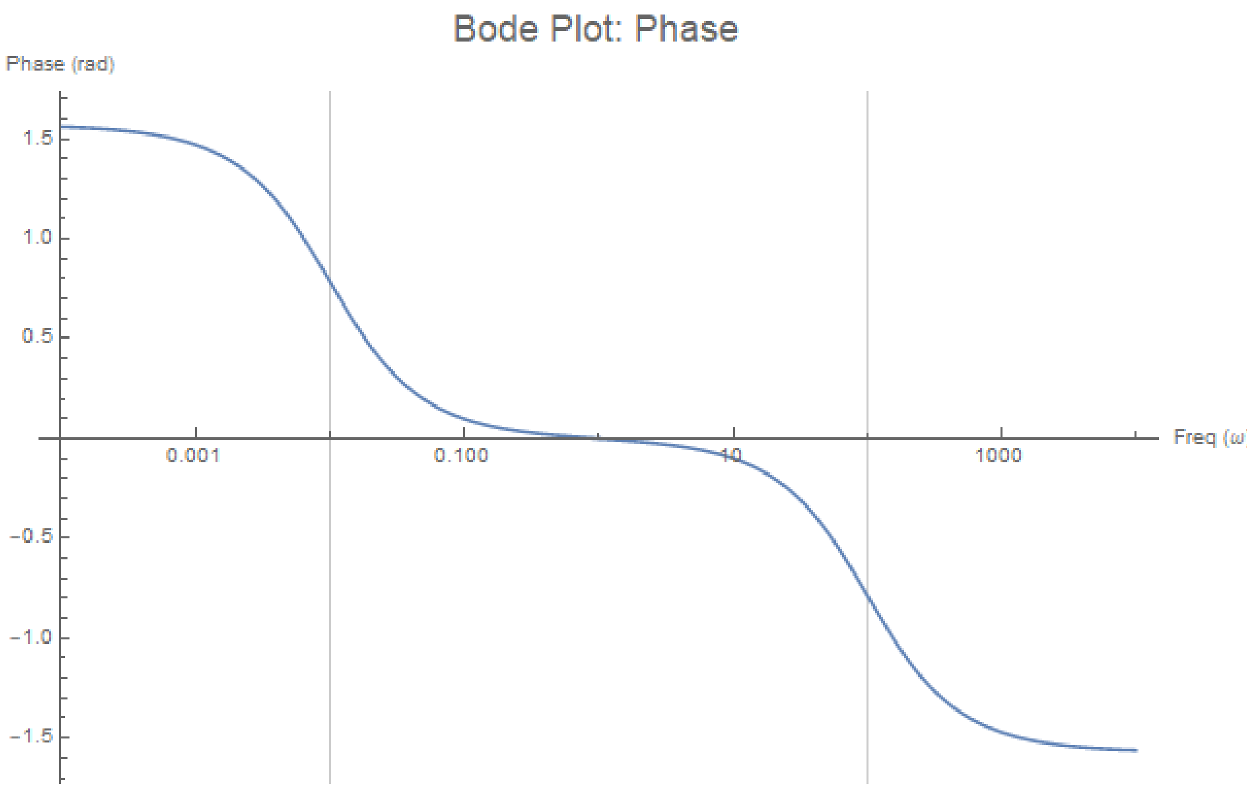

Figure 6: Bode plot. Values chosen: $R_f = 10$ kΩ and $C_f = 1$ μF, $R_{in} = 1$ MΩ and $C_{in} = 100$ μF.

Note that the regions where the phase drops is centered at the same break frequencies as found for magnitude. Otherwise, the phase shift remains constant. For the low frequency region, $G(\omega) \approx j\omega R_f C_{in}$, so $\angle G(\omega) \approx \arctan(\infty) = \pi/2$. Conversely, in the high frequency region, $G(\omega) \approx -j/\omega R_{in} C_f$, so $\angle G(\omega) \approx \arctan(-\infty) = -\pi/2$.