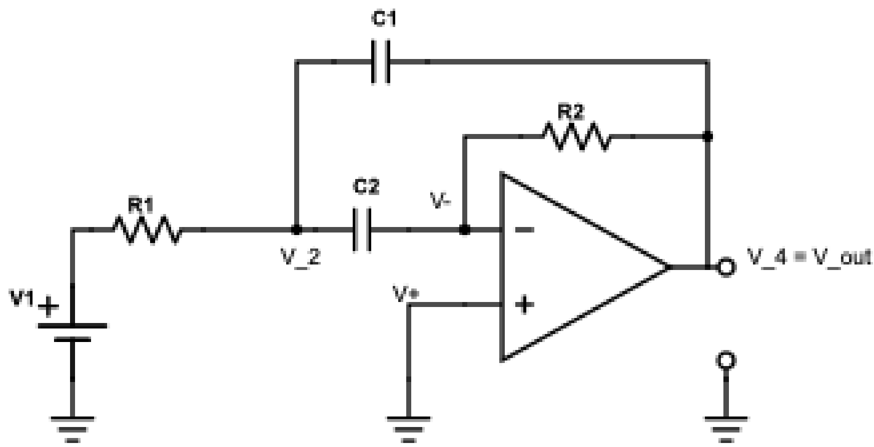


Figure 1: The circuit you are provided in this problem. Note that a voltage node, V_2, is defined for convenience and will be expressed in terms of V_1 and V_4 when we're done.

Problem 15.34a (BW).

We are told that $V_- = V_+$ in this problem as the Op Amp is ideal. This means that we can set $V_- = 0V$. By KCL, and the use of node V_2 as given in Figure 1, we can write the following relations (expressing everything in terms of general impedance to begin with, where Z_{C1} is the impedance of capacitor C_1 and Z_{C2} is the impedance of capacitor C_2

$$\frac{V_1 - V_2}{R_1} = \frac{V_2 - V_4}{Z_{C1}} + \frac{V_2 - V_4}{Z_{C2} + R_2} \tag{1}$$
$$= \frac{V_2 - V_4}{Z_{C1}} + \frac{V_2 - 0}{Z_{C2}} \tag{2}$$
$$= \frac{V_2 - V_4}{Z_{C1}} + \frac{V_2}{Z_{C2}} \tag{3}$$

To express V_2 in terms of values given in the problem, we solve another independent KCL relation:

$$\frac{V_2 - V_-}{Z_{C2}} = \frac{V_- - V_4}{R_2} \tag{4}$$
$$\frac{V_2}{Z_{C2}} = \frac{-V_4}{R_2} \tag{5}$$
$$V_2 = \frac{-V_4 * Z_{C2}}{R_2} \tag{6}$$

Plugging V_2 into (1) and (3) and solving for V_4 after some algebra (or Mathematica) yields:

$$\frac{V_1 + \frac{V_4 * Z_{C2}}{R_2}}{R_1} = \frac{\frac{-V_4 * Z_{C2}}{R_2} - V_4}{Z_{C1}} + \frac{\frac{-V_4 * Z_{C2}}{R_2}}{Z_{C2}} \tag{7}$$

After expansion of fractions by multiplying both sides by the denominators

$$V_1 * R_2 * Z_{C1} * Z_{C2} + V_4 * Z_{C2}^2 * Z_{C1} = -V_4 * Z_{C2}^2 * R_1 - V_4 * R_1 * Z_{C2} * R_2 - V_4 * Z_{C1} * Z_{C2} * R_1 \tag{8}$$

After solving for V_4/V_1 in terms of general impedances

$$\tag{9}$$

$$\frac{V_4}{V_1} = \boxed{\frac{-R_2 * Z_{C1}}{Z_{C1} * Z_{C2} + R_1 * (R_2 + Z_{C1} + Z_{C2})}} \tag{10}$$

At this point, plug in $Z_{C1} = \frac{1}{i\omega C_1}$ and $Z_{C2} = \frac{1}{i\omega C_2}$

$$\frac{V_4}{V_1} = \frac{-R_2 * \frac{1}{i\omega C_1}}{\frac{1}{i\omega C_1} * \frac{1}{i\omega C_2} + R_1 * \left(R_2 + \frac{1}{i\omega C_1} + \frac{1}{i\omega C_2}\right)} \tag{11}$$

Multiply both the numerator and denominator by $\frac{1}{i\omega C_1}$ to get

$$= \frac{-R_2}{i\omega C_1 \left(\frac{1}{\omega^2 C_1 C_2} + R_1 R_2 + \frac{R_1(C_1+C_2)}{i\omega C_1 C_2}\right)} \tag{12}$$

$$= \frac{-R_2 \omega C_2}{i + i\omega^2 R_1 R_2 C_1 C_2 + \omega R_1 (C_1 + C_2)} \tag{13}$$

Rationalize the fraction to get

$$\frac{V_4}{V_1} = \boxed{\frac{-\omega R_2 C_2 \left(\omega R_1 (C_1 + C_2) - i(1 + \omega^2 R_1 R_2 C_1 C_2)\right)}{\omega^2 R_1^2 \left(C_1 + C_2\right)^2 + \left(1 + \omega^2 R_1 R_2 C_1 C_2\right)^2}} \tag{14}$$

Problem 15.34b (BW). We can find the magnitude of expression (14), ($|V_4/V_1|$), to return something resembling the bottom expression:

$$|\frac{V_4}{V_1}| = \boxed{\frac{\omega R_2 C_2 \sqrt{\left(\omega^2 R_1^2 \left(C_1 + C_2\right)^2\right) + \left(1 + \omega^2 R_1 R_2 C_1 C_2\right)^2}}{\omega^2 R_1^2 \left(C_1 + C_2\right)^2 + \left(1 + \omega^2 R_1 R_2 C_1 C_2\right)^2}} \tag{15}$$

We can now make a log-log plot of the log of the magnitude of the output voltage and the input voltage vs the log of the frequency.

In[79]:=
$$f[\omega\_] := \frac{\omega \; r2 \; c2 \; \mathrm{Sqrt}\left[\left(\omega^2 \; r1^2 \; (c1+c2)^2\right) + \left(1+\omega^2 \; r2 \; r1 \; c1 \; c2\right)^2\right]}{\omega^2 \; r1^2 \; (c1+c2)^2 + \left(1+\omega^2 \; r1 \; r2 \; c1 \; c2\right)^2}$$

In[80]:= f[ω]

Out[80]=
$$\frac{c2 \; r2 \; \omega}{\sqrt{(c1+c2)^2 \; r1^2 \; \omega^2 + \left(1 + c1 \; c2 \; r1 \; r2 \; \omega^2\right)^2}}$$

```
In[81]:= c1 = 0.01*^-6;
         c2 = 0.01*^-6;
         r1 = 10;
         r2 = 1000;
```

In[85]:= f[ω]

Out[85]=
$$\frac{0.00001 \; \omega}{\sqrt{4. \times 10^{-14} \; \omega^2 + \left(1 + 1. \times 10^{-12} \; \omega^2\right)^2}}$$

In[76]:= LogLogPlot[f[ω], {ω, 1, 1*^12}, PlotRange → All]

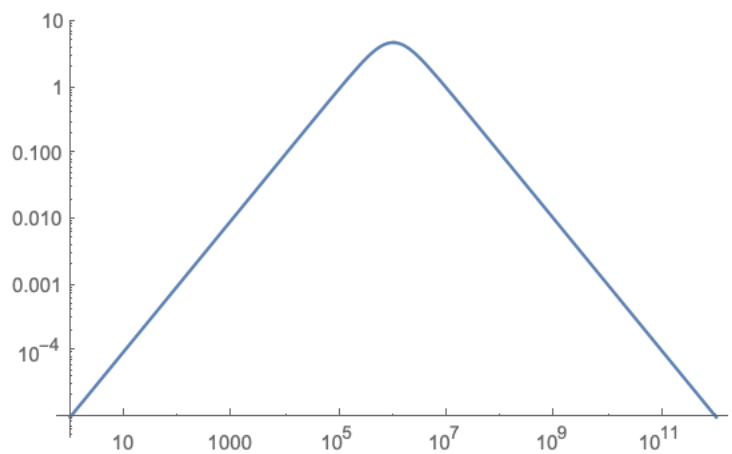
Out[76]=


Figure 2: Log-Log plot of $Log(|V_4/V_1|)$ on the y-axis vs $Log(\omega)$ on the x-axis

Problem 15.34c (BW). We can see from Figure 2 that the circuit shown in Figure 1 is a band pass filter. The resonance frequency of ω 10^6 is determined by solving for the frequency at which the denominator of (15) goes to zero (the magnitude blows up).

To solve for bandwidth using the definition given in the problem, we set the magnitude ratio equal to 1/2 and solve for the two values of ω.

$$\frac{1}{2} = \frac{\omega R_2 C_2 \sqrt{\left(\omega^2 R_1^2 \left(C_1 + C_2\right)^2\right) + \left(1 + \omega^2 R_1 R_2 C_1 C_2\right)^2}}{\omega^2 R_1^2 \left(C_1 + C_2\right)^2 + \left(1 + \omega^2 R_1 R_2 C_1 C_2\right)^2} \tag{16}$$

Solve using an awesome computer algebra system and take the two positive ω to get

$$\omega = \frac{\sqrt{2}}{[-\left(C_1 + C_2\right)^2 - 2C_1C_2R_1R_2 + 4C_2^2R_2^2 +} \tag{17}$$

$$\sqrt{\left(((C_1 + C_2)R_1 - 2C_2R_2)((C_1 + C_2)R_1 + 2C_2R_2)((C_1 + C_2)^2 R_1^2 + 4C_1C_2R_1R_2 - 4C_2^2R_2^2)\right)}]^{1/2} \quad 2005/06/28ver: 1.3s$$

and

$$\omega = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{C_1^2 C_2^2 R_1^2 R_2^2} * [-\left(C_1 + C_2\right)^2 R_1^2 - 2C_1C_2R_1R_2 + 4C_2^2R_2^2 +} \tag{19}$$

$$\left((C_1 + C_2)R_1 - 2C_2R_2)((C_1 + C_2)R_1 + 2C_2R_2)((C_1 + C_2)^2 R_1^2 + 4C_1C_2R_1R_2 - 4C_2^2R_2^2)\right)^{1/2}] \quad 2005/06/28ver: 1.3s$$

We find that the bandwidth is the magnitude of the difference between the two values of ω. Each ω is too long to fit on a line in (17) and (19) so they wrap around onto the next line.

Problem 15.35 (SH).

a)

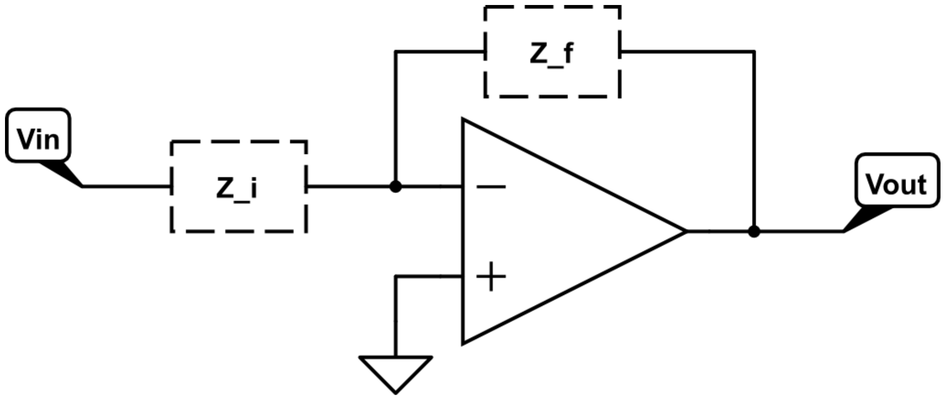


Figure 3: Op Amp with negative feedback using arbitrary impedances.

For arbitrary feedback impedance Z_f and input impedance Z_i, the transfer function is found by:

$$V_- \approx V_+ = 0 \qquad \text{Negative Feedback \& Large Gain}$$

$$\frac{V_{\text{out}} - V_-}{Z_f} = \frac{V_- - V_{\text{in}}}{Z_i} \qquad \text{KCL at negative terminal}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{Z_f}{Z_i} \qquad \text{Transfer function}$$

Here,

$$Z_f = R_2||C = \frac{R_2}{j\omega R_2 C + 1} \qquad Z_i = R_1 \tag{21}$$

Thus, our transfer function is simply:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1} \tag{22}$$

Using $R_2 = 10R_1$ as given in the problem statement,

$$\boxed{\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{-10}{10j\omega R_1 C + 1}} \tag{23}$$

Finding the magnitude and phase:

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{10}{\sqrt{100R_1^2C^2\omega^2 + 1}} \qquad \angle\frac{V_{\text{out}}}{V_{\text{in}}} = \arctan(-10R_1C\omega) \tag{24}$$

We can quickly see that if $\omega \ll \frac{1}{10R_1C}$, the 1 in the denominator will dominate the ω term. Let's see what this implies:

$$\omega \ll \frac{1}{10R_1C}:$$

$$\begin{aligned}
\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| &= 10\left(100R_1^2C^2\omega^2+1\right)^{-1/2} && \text{Rewrite}\\
&= 10\left[1+\left(\frac{\omega}{1/10R_1C}\right)^2\right]^{-1/2} && \text{Writing with } \omega \ll \frac{1}{10R_1C}\\
&\approx 10\left[1-\frac{1}{2}\left(10R_1C\omega\right)^2\right] && \text{Binomial Expansion}\\
&\approx 10 && \text{Keeping only up to first order in } \omega
\end{aligned}$$

Thus, for frequencies smaller than $\frac{1}{10R_1C}$, we see the magnitude is approximately constant. As you might expect, this approximation will change for frequencies that are larger than $\frac{1}{10R_1C}$; thus, it makes sense to refer to this as the break frequency, as it is where we will see a change in behavior in our magnitude of the Bode Plot.

For frequencies greater than the break frequency:

$$\omega \gg \frac{1}{10R_1C}:$$

$$\begin{aligned}
\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| &= 10\left(100R_1^2C^2\omega^2+1\right)^{-1/2} && \text{Rewrite}\\
&= \frac{1}{R_1C\omega}\left[1+\left(\frac{1}{10R_1C\omega}\right)^2\right]^{-1/2} && \text{Writing with } \frac{1}{10R_1C\omega} \ll 1\\
&\approx \frac{1}{R_1C\omega}\left[1-\frac{1}{2}\left(\frac{1}{10R_1C\omega}\right)^2\right] && \text{Binomial Expansion}\\
&\approx \frac{1}{R_1C\omega} && \text{Dropping higher order of } 1/\omega
\end{aligned}$$

We see that for high frequencies, the magnitude falls as $\propto \omega^{-1}$. For our Bode Plot, because we plot on Log Log scale for magnitude, we can see the plot should be flat before the break frequency as the function is approximately constant (slope is 0). After the break frequency, we should see a negative slope of -1, because $\log(\omega^{-1}) = -\log(\omega)$.

For the phase, we can quickly analyze this one without any problem. When $\omega \to 0$:

$$\angle\frac{V_{\text{out}}}{V_{\text{in}}} \approx \arctan(-10R_1C(0)) = \arctan(0) = 0$$

When $\omega \to \frac{1}{10R_1C}$:

$$\angle\frac{V_{\text{out}}}{V_{\text{in}}} = \arctan\left(-\frac{10R_1C}{10R_1C}\right) = \arctan(-1) = -\frac{\pi}{4}$$

Finally, when $\omega \to \infty$:

$$\angle\frac{V_{\text{out}}}{V_{\text{in}}} \approx \arctan(-10R_1C(\infty)) = \arctan(\infty) = -\frac{\pi}{2}$$

However, note that the transfer function has a negative constant. This means that we start off with a phase shift of π. Accounting for that, we would expect to see the phase at $-\pi$ for small frequencies and shift to $-3\pi/2$ for large frequencies. This makes sense, as this configuration is an active low-pass filter or an op-amp integrator. When the frequency is high enough, we expect the output to be the time integral of the input, multiplied by a -1. For a sine wave input, we would expect to see a cosine output, which is simply sine shifted forward by π (equivalent to shifting backwards by $3\pi/2$). We also know we must pass through $-5\pi/4$ as we transition from low to high frequencies. Piecing all of this together, the complete Bode Plot looks like the following with R_1C chosen to be 100

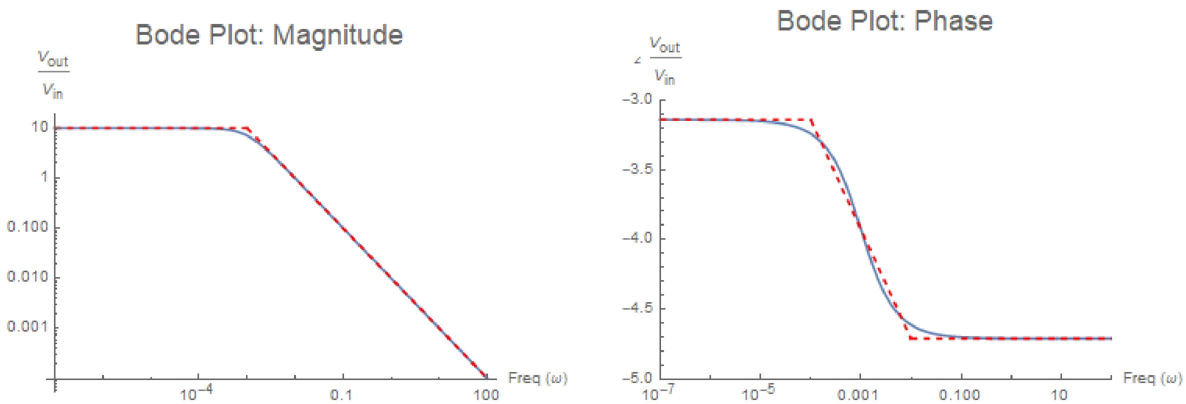


Figure 4: Bode Plot with $R_1C = 100$. Red is approximation, blue is actual plot.

b) Consider the Thevenin's equivalent at the input. The equivalent voltage is simply a voltage divider:

$$V_{th} = \frac{V_{\text{in}}}{j\omega R_1 C_x/2 + 1} = \frac{2}{j\omega R_1 C_x + 2} V_{\text{in}} \tag{25}$$

We find the equivalent impedance by shorting the "output" of the input circuit to ground. This places the negative terminal resistor in parallel with the capacitor.

$$Z_f = R_2 \qquad Z_{th} = \frac{R_1}{2} + C_x || \frac{R_1}{2} = \frac{R_1}{2}\left(\frac{j\omega R_1 C_x + 4}{j\omega R_1 C_x + 2}\right) \tag{26}$$

Thus, we can write:

$$\frac{V_{\text{out}}}{V_{th}} = \frac{V_{\text{out}}}{V_{\text{in}}}\frac{j\omega R_1 C_x + 2}{2} = -\frac{Z_f}{Z_{th}} = -\frac{R_2}{R_1}\frac{2j\omega R_1 C_x + 4}{j\omega R_1 C_x + 4} \tag{27}$$

Rearranging, we find our transfer function to be:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1}\frac{4}{j\omega R_1 C_x + 4} \tag{28}$$

Using $R_2 = 10R_1$, we can now equivocate this to our previously found transfer function in order to identify C_x:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{-40}{j\omega R_1 C_x + 4} = \frac{-10}{j\omega R_1 C + 1} \tag{29}$$

We therefore conclude:

$$\boxed{C_x = 4C} \tag{30}$$