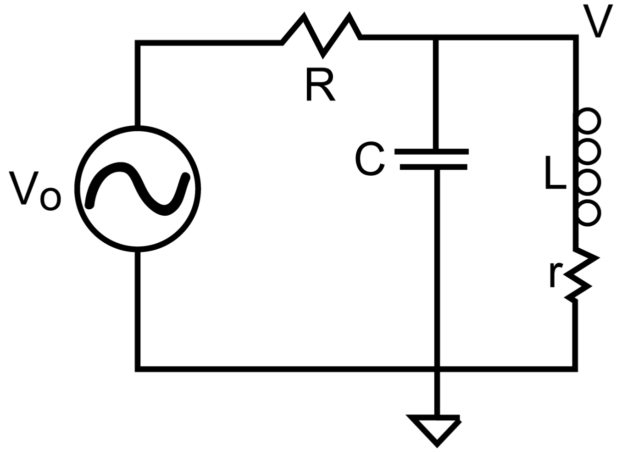
1 RLC resonator with a lossy inductor

The circuit in laboratory exercise 3.1 uses an indictor that has a modest resistance. Let's see how this effects the amplitude of the signal output.



The equivalent impedance of the (L+r)||C circuit is

$$
\begin{aligned}
Z_{eq} &= \frac{\frac{r+i\omega L}{i\omega C}}{r+i\omega L+\frac{1}{i\omega C}} \qquad (1.1)\\
&= \frac{r+i\omega L}{i+i\omega C(r+i\omega L)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{V(\omega)}{V_o(\omega)} &= \frac{\frac{r+i\omega L}{i+i\omega C(r+i\omega L)}}{\frac{r+i\omega L}{i+i\omega C(r+i\omega L)}+R} \qquad (1.2)\\
&= \frac{r+i\omega L}{r+i\omega L+R(1+i\omega C(r+i\omega L))}\\
&= \frac{r+i\omega L}{r+R-R\omega^2 LC+i\omega L+i\omega rRC}.
\end{aligned}
$$

We let

$$\omega_0 = \frac{1}{\sqrt{LC}}, \qquad (1.3)$$

$$\rho = r/R, \qquad (1.4)$$

where the limit of a lossless inductor then corresponds to $\rho \to 0$, and

$$\tau = L/R. \qquad (1.5)$$

In terms of these normalized units,

$$\frac{V(\omega)}{V_o(\omega)} = \frac{\rho+i\omega\tau}{1+\rho-\left(\frac{\omega}{\omega_0}\right)^2+i\omega\tau\left(1+\frac{\rho}{(\omega_0\tau)^2}\right)}. \qquad (1.6)$$

The magnitude of the output is then

$$\left|\frac{V(\omega)}{V_o(\omega)}\right| = \sqrt{\frac{\rho^2+(\omega\tau)^2}{\left(1+\rho-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+(\omega\tau)^2\left(1+\frac{\rho}{(\omega_0\tau)^2}\right)^2}} \tag{1.7}$$

$$= \sqrt{\frac{(\frac{\omega}{\omega_0})^2+(\frac{\rho}{\omega_0\tau})^2}{\left(\frac{1-\left(\frac{\omega}{\omega_0}\right)^2+\rho}{\omega\tau}\right)^2+(\frac{\omega}{\omega_0})^2\left(1+\frac{\rho}{(\omega_0\tau)^2}\right)^2}}$$

and this has the expected limit as $\rho \to 0$, i.e.,

$$\left|\frac{V(\omega)}{V_o(\omega)}\right| \xrightarrow{\rho \to 0} \frac{\frac{\omega}{\omega_0}}{\sqrt{\left(\frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\omega\tau}\right)^2+(\frac{\omega}{\omega_0})^2}} \tag{1.8}$$

We return to the general expression of the magnitude of the output and note that for $\omega = \omega_0$, the magnitude of the output is

$$\left|\frac{V(\omega)}{V_o(\omega)}\right| \xrightarrow{\omega = \omega_0} \sqrt{\frac{1+\left(\frac{\rho}{\omega_0\tau}\right)^2}{\left(\frac{\rho}{\omega_0\tau}\right)^2+\left(1+\frac{1}{\rho}\left(\frac{\rho}{\omega_0\tau}\right)^2\right)^2}}. \tag{1.9}$$

The dependence on $\frac{\rho}{\omega_0\tau} = r\sqrt{\frac{C}{L}}$ shows that loss in the inductors is minimized at higher frequencies.

For the components in laboratory 3.1, i.e., $L = 10$ mH, $R = 100$ kΩ, $C = 0.01$ μF and r measured to be 150 Ω for one inductor, we find $\rho = 0.015$, $\omega\tau = 0.01$, and the decrement in amplitude is large, a factor of 16. This occurs even through ρ is rather small.