1 Notes on time domain circuit analysis

1.1 Background

The most common first-order differential equation in circuit analysis, such as for a RC high-pass or low-pass filter, has constant coefficients and is of the form

$$\tau \frac{dV(t)}{dt} + V(t) = F(t). \tag{1.1}$$

The full solution is the solution to the homogeneous equation, denoted $V_H(t)$ plus the solution to the inhomogeneous solution, denoted $V_I(t)$. The homogeneous equation is the part with $F(t) = 0$, i.e., the equation for the decay from $V(0)$ for which the homogeneous solution is, to within a constant,

$$V_H(t) = V(0^+)\Phi(t) \tag{1.2}$$

with

$$\Phi(t) = e^{-t/\tau}. \tag{1.3}$$

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The inhomogeneous solution satisfies the full equation

$$\tau \frac{dV_I(t)}{dt} + V_I(t) = F(t) \tag{1.4}$$

and we take as an ansatz that the inhomogeneous solution is proportion to the homogeneous solution times a yet to be detemined function of time $\alpha(t)$, i.e.,

$$\begin{aligned} V_I(t) &= \Phi(t)\alpha(t) \\ &= e^{-t/\tau}\alpha(t) \end{aligned} \tag{1.5}$$

and

$$\tau \frac{dV_I(t)}{dt} = \tau e^{-t/\tau} \frac{d\alpha(t)}{dt} - \tau e^{-t/\tau} \frac{1}{\tau}\alpha(t). \tag{1.6}$$

If we plug these back into the original equation, we get

$$\tau e^{-t/\tau} \frac{d\alpha(t)}{dt} - e^{-t/\tau}\alpha(t) + e^{-t/\tau}\alpha(t) = F(t) \tag{1.7}$$

or

$$\frac{d\alpha(t)}{dt} = \frac{1}{\tau} e^{t/\tau} F(t) \tag{1.8}$$

or

$$\alpha(t) = \int_0^t \frac{dt'}{\tau} e^{t'/\tau}\, F(t') + \text{Constant} \tag{1.9}$$

and, substituting back for $\alpha(t)$ gives

$$V_I(t) = e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \, F(t') \; + \; \text{Constant} \tag{1.10}$$

$$= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} \, F(t') \; + \; \text{Constant}.$$

Thus

$$V(t) = V_I(t) \; + \; V_H(t) \tag{1.11}$$

$$= \int_0^t \frac{dt'}{\tau} \, e^{-(t-t')/\tau} \, F(t') \; + \; V(0^+) e^{-t/\tau}$$

The integral for the driven response is call the "convolution" integral. The above relation is a special case, for constant coefficients, between the input and the driven response for a linear system. In general, the response of such a linear system to a pulse ("delta function") is given by $\Phi(t)$ and

$$V_I(t) = \int_0^t \frac{dt'}{\tau} \, \Phi(t-t') \, F(t'). \tag{1.12}$$

1.2 Example of step input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0$ for $t \geq 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} \, e^{-(t-t')/\tau} \, V_0 \; + \; V(0^+) e^{-t/\tau} \tag{1.13}$$

$$= V_0 \, (1 \; - \; e^{-t/\tau}) \; + \; V(0^+) e^{-t/\tau}$$

1.3 Example of sine input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0 sin(\omega_0 t)$ for $t \geq 0$. For focus only on the driven part and take $V(0^-) = 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} \, e^{-(t-t')/\tau} \, V_0 sin(\omega_0 t') \tag{1.14}$$

$$= V_0 \, e^{-t/\tau} \int_0^t \frac{dt'}{\tau} \, e^{t'/\tau} \, \frac{e^{i\omega_0 t'} - e^{-i\omega_0 t'}}{2i}$$

$$= V_0 \, \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx \, e^x \left(e^{i\omega_0 \tau x} - e^{-i\omega_0 \tau x} \right)$$

$$
\begin{aligned}
&= V_0 \, \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx \, \left(e^{(1+i\omega_0\tau)x} - e^{(1-i\omega_0\tau)x}\right) \\
&= V_0 \, \frac{e^{-t/\tau}}{2i} \, \left(\frac{e^{t/\tau}e^{i\omega_0 t} - 1}{1+i\omega_0\tau} - \frac{e^{t/\tau}e^{-i\omega_0 t} - 1}{1-i\omega_0\tau}\right) \\
&= V_0 \, \frac{1}{2i} \, \left[\left(\frac{e^{i\omega_0 t}}{1+i\omega_0\tau} - \frac{e^{-i\omega_0 t}}{1-i\omega_0\tau}\right) - e^{-t/\tau}\left(\frac{1}{1+i\omega_0\tau} - \frac{1}{1-i\omega_0\tau}\right)\right] \\
&= V_0 \, \frac{1}{2i} \, \left[\frac{(1-i\omega_0\tau)e^{i\omega_0 t} - (1+i\omega_0\tau)e^{-i\omega_0 t} + 2i\omega_0\tau e^{-t/\tau}}{1+(\omega_0\tau)^2}\right] \\
&= V_0 \, \frac{1}{1+(\omega_0\tau)^2} \, \left[\frac{e^{i\omega_0 t} - e^{i\omega_0 t}}{2i} - \omega_0\tau\frac{e^{i\omega_0 t} + e^{i\omega_0 t}}{2} + \omega_0\tau e^{-t/\tau}\right] \\
&= V_0 \, \left[\frac{1}{1+(\omega_0\tau)^2}\,\sin(\omega_0\mathrm{t}) - \frac{\omega_0\tau}{1+(\omega_0\tau)^2}\,\cos(\omega_0\mathrm{t}) + \frac{\omega_0\tau}{1+(\omega_0\tau)^2}\,\mathrm{e}^{-\mathrm{t}/\tau}\right]
\end{aligned}
$$

The sine term represents a faithful transmission of the input and the cosine term represents a phase shifted version, while the exponential represents the transient from turning the signal on at t=0. With a bit more algebra, and recalling that $\sin(a-b) = \sin(a)\cos(b)$ - $\cos(a)\sin(b)$, we can simply this.

$$
\begin{aligned}
V(t) &= \frac{V_0}{\sqrt{1+(\omega_0\tau)^2}} \left[\frac{1}{\sqrt{1+(\omega_0\tau)^2}}\sin(\omega_0\mathrm{t}) - \frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}}\cos(\omega_0\mathrm{t}) + \frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}}\mathrm{e}^{-\mathrm{t}/\tau}\right] \\
&= \frac{V_0}{\sqrt{1+(\omega_0\tau)^2}} \left[\sin\left[\omega_0\mathrm{t} - \mathrm{atan}(\omega_0\tau)\right] + \frac{\omega_0\tau}{\sqrt{1+(\omega_0\tau)^2}}\mathrm{e}^{-\mathrm{t}/\tau}\right] \qquad (1.15)
\end{aligned}
$$

At short times, i.e., $t << \tau$,

$$
V(t) \approx \frac{\omega_0\tau}{1+(\omega_0\tau)^2}\,\frac{t}{\tau} \qquad (1.16)
$$

and in steady state, i.e., $t >> \tau$,

$$
V(t) = \frac{V_0}{\sqrt{1+(\omega_0\tau)^2}}\,\sin\left[\omega_0\mathrm{t} - \mathrm{atan}(\omega_0\tau)\right] \qquad (1.17)
$$