

# Notes on Fourier Transform,

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DK 25 FEB 11

Let us review some with:

$$V(t) \xrightarrow{\text{Fourier}} \tilde{V}(\omega)$$

$$\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt V(t) e^{i\omega t}$$

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \tilde{V}(\omega) e^{-i\omega t}$$

$$" = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dx V(x) e^{i\omega(x-t)}$$

$$" = \int_{-\infty}^{+\infty} dx V(x) \underbrace{\left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega(x-t)} \right]}_{\delta(x-t)}$$

Let us look @ correlation

$$C(\tau) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt V(t) W(t-\tau)$$

$$\tilde{C}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\tau C(\tau) e^{i\omega\tau}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dt V(t) W(t-\tau) e^{i\omega\tau}$$

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$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dt V(t) W(t-\tau) e^{-i\omega(t-\tau)} e^{+i\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt V(t) e^{+i\omega t} \int_{-\infty}^{+\infty} d\tau W(t-\tau) e^{-i\omega(t-\tau)}$$

$$x = t - \tau$$

$$dx = -d\tau$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt V(t) e^{+i\omega t}}_{\tilde{V}(\omega)} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx W(x) e^{-i\omega x}}_{\tilde{W}^*(\omega)}$$

$$= \tilde{V}(\omega) \tilde{W}^*(\omega)$$

Part - Conservation of energy

$$\int_{-\infty}^{+\infty} dt V^2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \tilde{V}(\omega) e^{-i\omega t} \int_{-\infty}^{+\infty} dx \tilde{V}(x) e^{+ixt}$$

$$= \int_{-\infty}^{+\infty} dx V(x) \int_{-\infty}^{+\infty} d\omega \tilde{V}(\omega) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-i(\omega+x)t}$$

$$= \int_{-\infty}^{+\infty} d\omega \tilde{V}(\omega) \tilde{V}(-\omega) \underbrace{\int_{-\infty}^{+\infty} dt e^{-i(\omega+x)t}}_{\delta(\omega+x)}$$

$$\text{but } \tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt V(t) e^{-i\omega t} \quad \tilde{V}(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt V(t) e^{+i\omega t}$$

$$\therefore \int_{-\infty}^{+\infty} dt V^2(t) = \int_{-\infty}^{+\infty} du \tilde{V}(u) \tilde{V}^*(u)$$

$$= \int_{-\infty}^{+\infty} du |\tilde{V}(u)|^2$$

↑  
Power spectrum

Energy is conserved as integrated power is the same in either coordinate system.

Transform pairs.

$$V(t) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}$$

$$\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} dt e^{+i\omega t} e^{-t^2/2\sigma^2}$$

$$e^{-\frac{1}{2\sigma^2}(t^2 - 2\sigma^2 i\omega t + \omega^2 \sigma^4)}$$

$$\times e^{-\frac{\omega^2 \sigma^2}{2}}$$

$$\equiv \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2(\sigma^2)^2} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} dt e^{-\frac{1}{2\sigma^2}(t - i\omega\sigma^2)^2}}_1$$

↑  
Variance =  $1/\sigma^2$

Look @ convolution (cont  $\rightarrow \infty$ )

$$X(t) = \int_{-\infty}^t dx R(t-x) S(x)$$

$$\tilde{V}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \int_{-\infty}^t dx R(t-x) S(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx S(x) \int_{-\infty}^{+\infty} dt e^{i\omega t} R(t-x) \quad \begin{array}{l} y = t-x \\ t = y+x \\ dy = dt \end{array}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^t dx S(x)}_{\substack{t \rightarrow \infty \\ \tilde{S}(\omega)}} e^{i\omega x} \underbrace{\int_{-\infty}^{+\infty} dy e^{i\omega y} R(y)}_{\tilde{R}(\omega)}$$

$$\tilde{V}(\omega) = \tilde{S}(\omega) \tilde{R}(\omega)$$