Phys 178: HW2

February 2024

Due midnight on Feb. 14th (no exceptions this time!). Please justify all of your answers and submit to Gradescope. Make sure you select the correct question for each part of your submission. If you have any questions, please email Ghita Guessous (gguessou@ucsd.edu).

- All physics students (graduate and undergrad) are required to do both Problems.
- Undergraduate biology students are required to do Problems 1.1 and 2 but are encouraged to attempt the rest

1 Ring Attractor Model

In this question, we investigate the phase diagram of the ring attractor model discussed in class.

$$\frac{dr(\phi,t)}{dt} = -r(\phi,t) + [W_0 r_0(t) + W_1 r_1(t)\cos(\phi - \psi(t)) - \theta]_+, \tag{1}$$

where $r_0(t)$, $r_1(t)$ and $\psi(t)$ are from the Fourier series of $r(\phi,t)$,

$$r(\phi,t) = r_0(t) + 2r_1(t)\cos(\phi - \psi(t)) + \sum_{|n|\geq 2} r_n(t)e^{in\phi}.$$

and $W_1 > 0$ for local excitation. Note that in Eq. (1), the input is assumed to have very weak modulation $I_1 \approx 0$ and the background input I_0 can be incorporated in the threshold parameter θ. For simplicity, we further assume $\theta < 0$.

1.1 Steady state of the ring attactor model

The steady state/fixed point of Eq. (1) is given by

$$r(\phi) = [W_0 r_0 + W_1 r_1 \cos(\phi - \psi) - \theta]_+. \tag{2}$$

Note that by re-defining $r(\phi) \equiv r(\phi + \psi)$ or $r(\phi + \psi + \pi)$, we can take $\psi = 0$ and $r_1 \geq 0$ without loss of generality. Following similar ideas as in class, **find** the conditions on (W_0, W_1) such that the steady state profile $r(\phi)$ satisfies

a. (Homogeneous state) $W_0 r_0 + W_1 r_1 \cos\phi - \theta \geq 0$, for all $\phi \in [0, \pi]$.

b. (Bump state) $W_0 r_0 + W_1 r_1 \cos\phi_C - \theta = 0$, for some $\phi_C \in [0, \pi)$.

1.2 Stability of the steady states

We assume a small perturbation around the steady state $r(\phi,t) = r(\phi) + \delta r(\phi,t)$. By Eq. (1), the perturbations in the Fourier coefficients satisfy

$$\frac{d\delta r_0(t)}{dt} = -\delta r_0(t) + \left[\frac{W_0}{2\pi}\int_{-\pi}^{\pi}\Theta(r(\phi))\,d\phi\right]\delta r_0(t) + \left[\frac{W_1}{2\pi}\int_{-\pi}^{\pi}\Theta(r(\phi))\cos\phi\,d\phi\right]\delta r_1(t),$$
$$\frac{d\delta r_1(t)}{dt} = -\delta r_1(t) + \left[\frac{W_0}{2\pi}\int_{-\pi}^{\pi}\Theta(r(\phi))\cos\phi\,d\phi\right]\delta r_0(t) + \left[\frac{W_1}{2\pi}\int_{-\pi}^{\pi}\Theta(r(\phi))\cos^2\phi\,d\phi\right]\delta r_1(t), \qquad (3)$$

where $\Theta(x)$ is the Heaviside step function, and

$$\frac{d\delta\psi(t)}{dt} = 0,$$
$$\frac{d\delta r_n(t)}{dt} = f_n(\delta r_0(t), \delta r_1(t), \delta\psi(t)), \quad \text{for} \quad |n| \geq 2. \qquad (4)$$

[If you are interested, try to show the above equations yourself.] Convince yourself that we only need to focus on Eq. (3) to understand the stability under small perturbation.

Find the regions in (W_0, W_1)-plane such that

a. The homogeneous state in 1.1a is stable.

b. The bump state in 1.1b is stable.

[Hint: You can just plot the regions numerically.]

1.3 Moving bump with weakly modulated inputs

Suppose (W_0, W_1) are in the region where the bump state is stable. Initially, the network is in a bump state, $r(\phi)$, peaked at $\phi = 0$ as in 1.1b. Starting at time $t = 0$, the network receives a modulated input $I_1(\phi,t) = \epsilon\cos(\phi - \phi_0)$, with small amplitude $0 < \epsilon \ll 1$ and peak at $\phi_0 \neq 0$. Such input will induce changes in the network activity $\delta r(\phi,t)$.

a. It can be shown from Eq. (3) that the resulted changes in the Fourier coefficents $\delta r_0(\phi,t)$ and $\delta r_1(\phi,t)$ will vanish at $t \to +\infty$. However, the change in $\psi(t)$ satisfies (up to the first order of ϵ)

$$\frac{d\psi(t)}{dt} = \frac{\epsilon}{2\pi}\int_{\psi(t)-\phi_C}^{\psi(t)+\phi_C}\sin(\phi - \psi(t))\cos(\phi - \phi_0)\,d\phi. \qquad (5)$$

where ϕ_C is the angle in 1.1b and is independent of t.

Show that $\psi(t) \to \phi_0$ as $t \to +\infty$, by completing the integral in the above equation.

b. We investigate the change of network activity numerically. Consider N neurons whose firing rates are described by

$$\frac{dr_i(t)}{dt} = -r_i(t) + f\left[\frac{1}{N}\sum_{j=1}^{N} W_{ij} r_i(t) + I_1\cos(\phi_i - \phi_0(t))\right], \quad \text{for} \quad i = 1, 2, ..., N,$$

where $\phi_i = \frac{2\pi}{N}i$ and $W_{ij} = W_0 + W_1\cos(\phi_i - \phi_j)$. For convenience, we use $f(x) = \frac{1+\tanh x}{2}$. Let $W_0 = -6$, $W_1 = 10$, $N = 500$ and $I_1 = 0.02$. Design the following input

$$\phi_0(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq t_b, \\ 2\pi/3 & \text{if} \quad t > t_b, \end{cases}$$

with $t_b = 500$. **Plot** the peak position of the network activity as a function of time.

2- Ring Attractor Numerics

In this problem, we will numerically simulate the dynamics of the ring attractor.

Warm up: exploring the properties of the weight matrix

- First create a network consisting of N=1000 neurons, that are evenly spaced along a ring with each neuron having a preferred angle $\phi_i = i\ (2\pi/N)$ (Eq. 3.4)
- Create the angle distance matrix $W(\phi_i, \phi_j) = W(\phi_i - \phi_j) = W(\Delta\phi)$
- We can Fourier expand the matrix as $W(\Delta\phi) = W_0 + W_1 \cos(\Delta\phi)$ (Eq.3.9). Let's consider two cases:
  - $W1 > 0, W_0 = 0$
  - $W1 > 0, W_0 = \alpha W_1$ where $1 < \alpha < 2$

  Plot the synaptic weights $W(\Delta\phi)$, where $\Delta\phi \in [-\pi, \pi]$. Which values of $\Delta\phi$ correspond to local vs. distal neurons? What do the weights reveal about connections between local and distal neurons? Are the connections inhibitory or excitory? How does W_0 affect the weights?
- Now add a bias term ϕ_{bias} such that $W(\Delta\phi) = W_0 + W_1 \cos(\Delta\phi - \phi_{bias})$. What changes?

Response to static external stimulus

- Set up the connection matrix W, with $W1 > 0$, $W_0 = 0$, $\phi_{bias} = 0$
- Set up a static external stimulus (i.e: none of the parameters are time dependent) such that $I(\Delta\phi, t) = I(\Delta\phi)$ (Eq. 3.11)
$$I(\phi - \phi_0, t) = I(\Delta\phi, t) = I_0(1+\epsilon) + I_1\epsilon\cos(\phi - \phi_0)$$
- Solve the corresponding ODE (Eq. 3.13) for the static stimulus defined above
$$\tau\frac{\mathrm{d}}{\mathrm{dt}}r_i + r_i(t) = f\left[\Sigma_{j=1}^{N} W_{ij} r_j + I(\Delta\phi)\right]$$
- Plot the firing rate of the ith neuron, $r_i(t)$ as a function of the preferred angle ϕ_i as well as the steady-state activity profile $r_i(t = t_{end}) = \overline{r_i}$

Response to dynamic external stimulus

- Set up the connection matrix W, with $W1 > 0$, $W_0 = 0$, $\phi_{bias} = 0$
- This time, set up a dynamic external stimulus (Eq. 3.11)
$$I(\Delta\phi, t) = I_1\epsilon\cos(\phi - \phi_0(t))$$
  Where
$$\phi_0(t) = \begin{cases} \phi_0 + \dfrac{\pi}{4}\ for\ t < t_b \\ \phi_0 - \dfrac{\pi}{4} for\ t \geq t_b \end{cases}$$
- Solve the corresponding ODE (Eq. 3.13) for the stimulus defined above:
$$\tau\frac{\mathrm{d}}{\mathrm{dt}}r_i + r_i(t) = f\left[\Sigma_{j=1}^{N} W_{ij} r_j + I(\Delta\phi, t)\right]$$
- Plot the firing rate of the ith neuron, $r_i(t)$ as a function of the preferred angle ϕ_i.