# PHYS 178/278: HW1

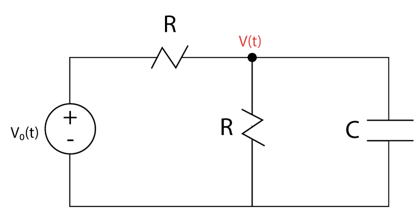
January 19, 2024

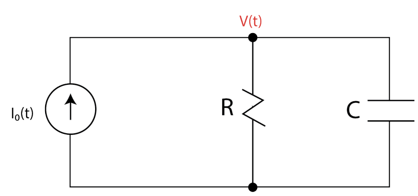
Due midnight on Jan. 29th. Please justify all of your answers and submit to Gradescope. Make sure you select the correct question for each part of your submission. If you have any questions, please email Ghita Guessous (gguessou@ucsd.edu).

- All physics students (graduate and undergrad) are required to do Problems 4 and 5, but can choose to also do Problems 1,2 and 3.
- Undergraduate biology students are required to do Problems 1,2 and 3 but are encouraged to attempt the rest
- Everyone else is required to choose three problems to do

## 1 Circuits

Solve for $V(t)$ for each of the two circuits below.




In the first circuit:

$$V_0(t) = \begin{cases} 0, & \text{for } t < 0 \\ V_0, & \text{for } t \geq 0 \end{cases}$$

In the second circuit:

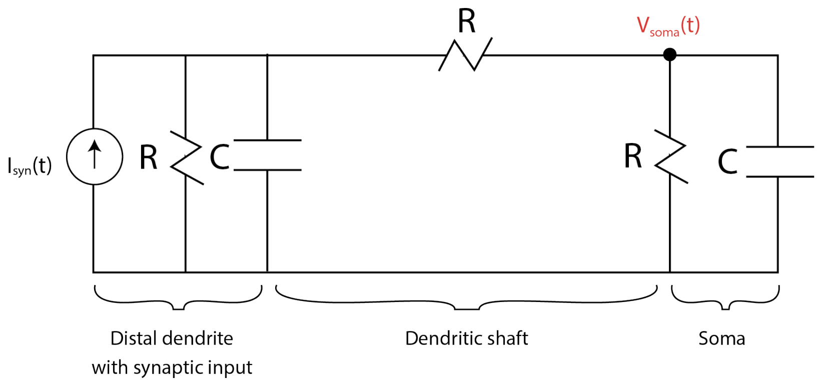
$$I_0(t) = \begin{cases} 0, & \text{for } t < 0 \\ I_0, & \text{for } 0 \leq t < a \\ 0 & \text{for } t \geq a \end{cases}$$

## 2 Circuit representation of a neuron

A neuron with a long dendrite can be modeled by two compartments, one for the soma and one for the dendrite. This model involves two coupled linear equations. This model is represented by the circuit below where:

$$I_{syn}(t) = \begin{cases} 0, & \text{for } t < 0 \\ I_0, & \text{for } 0 \leq t < a \\ 0 & \text{for } t \geq a \end{cases}$$

Solve for $V_{soma}(t)$.



## 3 A Hopfield model

Numerically investigate the storage capacity of the Hopfield network.

- Build a N = 400 neuron network.
- Construct enough stored states, $\vec{\xi}^{\mu}$, to satisfy $P/N = 0.20$, i.e., well above the expected capacity limit $\alpha = 0.14$ (Eq. 2.31).
- Choose each element $\vec{\xi}_i^{\nu}$ of the P = 0.2 × 400 = 80 patterns at random.
- Construct the weight matrix $W_{i,j}$ for storing one pattern (Eq. 2.6). Test, by recurrent action (Eq. 2.3), if the Hopfield model with one stored pattern maintains that pattern as a stable state*.
- Construct the weight matrix $W_{i,j}$ with two stored patterns (Eq 2.7). Test, by recurrent action (Eq. 2.3), if the Hopfield model maintains both patterns as stable states*.
- Continue this exercise all the way up to fifty stored patterns.
- Plot the average, fractional error in recall as a function of $P/N$ for $P = 1$ to $P = 80$.

* The error for each pattern is best calculated as the number of outputs, after recurrent action (Eq. 2.3) has reached a steady state, as the different from the final state $\vec{S}(t \to \infty)$ and the pattern $\vec{\xi}_i^{\nu}$. Thus the average, fractional error is:

$$\frac{1}{P}\sum_{\nu=1}^{P}\frac{1}{N}\sum_{i=1}^{N}\left|\frac{S_i(t\to\infty)-\vec{\xi}_i^{\nu}}{2}\right|$$

What is you best estimate of the storage capacity (Eq. 2.31) from your analysis? Suppose you repeat this exercise with newly chosen random vectors. What aspect of you estimate will improve? What aspect will still be problematic? (If you like, you can visualize the output and stored patterns by reformatting the 4001 vector $\vec{S}$ as a 2020 array).

# 4 A circuit of two binary neurons with mutual inhibition

We investigate the dynamics of a two-neuron circuit with mutual inhibitions. This is a common motif underlying many phenomena. The dynamics of the two neurons are given by:

$$\begin{aligned} S_1^{t+1} &= \operatorname{sign}(W_2 S_2^t - \theta_1) \\ S_2^{t+1} &= \operatorname{sign}(W_1 S_1^t - \theta_2) \end{aligned} \tag{1}$$

where $W_1 < 0$ and $W_2 < 0$,

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ -1 & \text{if} \quad x < 0, \\ \text{no changes} & \text{if} \quad x = 0, \end{cases} \tag{2}$$

and each neuron can be in $\pm 1$ state. The updating rule is

$$S_i^{t+1} = \begin{cases} 1 & \text{if } W_i S_i^t > \theta_2, \\ -1 & \text{if } W_i S_i^t < \theta_2, \\ S_i^t & \text{if } W_i S_i^t = \theta_2. \end{cases} \tag{3}$$

where $S_i^t \in \{-1, +1\}$ is the state of neuron $i$ at time $t$. The same relation holds for both indices $i = 1, 2$. Note that the connections are asymmetric if $W_1 \neq W_2$. We will fix the strength of inhibitory connections $W_{1,2}$ and analyze the dynamics for different values of $(\theta_1, \theta_2)$, which can be tuned by external current inputs.

The network $(S_1, S_2)$ has four possible states: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$ and $(-1, -1)$.

**1** **Find** the conditions on $(\theta_1, \theta_2)$ such that the following state is the fixed point of Eq.(1)

- **a.** $(+1, +1)$
- **b.** $(+1, -1)$
- **c.** $(-1, +1)$
- **d.** $(-1, -1)$

**2** Based on the above results, you should be able to identify five regions (ignore the boundary lines between them) with different fixed points: I. $(+1, +1)$ is the only fixed point; II. $(-1, +1)$ is the only fixed point; III. $(+1, -1)$ is the only fixed point; IV. $(-1, -1)$ is the only fixed point; V. Both $(-1, +1)$ and $(+1, -1)$ are fixed points.

- **a.** Sketch these regions in a phase diagram with $(\theta_1, \theta_2)$ as the coordinates.
- **b.** In region I-IV, **describe** what happens if starting from a state that is not a fixed point.
- **c.** In region V, **describe** what happens if starting from a state that is not a fixed point.

# 5 Binary neurons with symmetric connections and Hopfield model

## 5.1 Consider the generalized dynamics of a network of binary neurons

We have

$$S_i^{t+1} = \begin{cases} 1 & \text{if} \quad \sum_{j\neq i} W_{ij} S_j^t > \theta_i, \\ -1 & \text{if} \quad \sum_{j\neq i} W_{ij} S_j^t < \theta_i, \\ S_i^t & \text{if} \quad \sum_{j\neq i} W_{ij} S_j^t = \theta_i. \end{cases} \tag{4}$$

where $S_i^t \in \{-1, +1\}$ is the state of neuron $i$ at time $t$ and $i, j = 1, 2, 3..N$. We assume all the connections are symmetric $W_{ij} = W_{ji}$ and there are no self-connections $W_{ii} = 0$ for all $i, j$'s. The updating rule is asynchronous; in the physics literature, this is usually called Glauber dynamics at zero temperature. At each time step, select at random a neuron $i$ to be updated with equal probabilities $\forall i$.

In class, we have shown the dynamics given by Eq.(4) has a energy function (Lyapunov function)

$$E = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i} S_i W_{ij} S_j + \sum_{i=1}^{N} \theta_i S_i. \tag{5}$$

Specifically, when the states of neurons are updated, the energy is non-increasing.

### 5.1.1 Predicting neuronal dynamics from the energies of the states

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**a** Suppose at time step $t$, neuron $i$ is selected to be updated according to Eq.(4). **Show** that it will be changed, i.e., $S_i^{t+1} = -S_i^t$, if and only if the Energy after the change is *strictly* less than the energy without a change.
[Hint: The change of energy $\Delta E = -(\mu_i - \theta_i)\Delta S_i \leq 0$. What happens if $\Delta E = 0$ and $\Delta E < 0$?]

**b** A state $\vec{S} = (S_1, S_2, ...S_N)$ is called a *fixed point* of Eq.(4) if the state of every neuron $S_i$ won't be changed by Eq.(4). A state is called a *local minimum* of the energy function in Eq.(5) if the change of energy is non-negative ($\Delta E \geq 0$) after changing any one of the neurons. Using the result in **a**, show that a state $\vec{S}$ is a fixed point of Eq.(4) if and only if it is a local minimum of the energy function in Eq.(5).
[Hint: Show both directions.]

**c** Using the results in **a** and **b**, **show** that at long time $t \to +\infty$, the network will always reach one of the fixed points of Eq.(4), namely, the local minimum of Eq.(5), irrespective of the initial state.
[Hint: Recall that the system only has a finite number ($2^N$) of states.]

### 5.1.2 We investigate the behavior of Hopfield model (Eq.(6)) numerically

We take a network of $N = 900$ neurons; this can represent a 30×30 picture. The set of pictures are selected from a "bank" of 60 pictures (in the file `memory.mat`, see Fig. 1). We will look how the performance of the network changes as we include more and more memories/pictures.

**a** Choose your favorite picture in the bank and set the weight to include only this single pattern $W_{ij} = \xi_i \xi_j$. Start from a random initial condition and run the dynamics in Eq.(4) until the states in two consecutive steps are identical. **Plot** the picture you choose, the initial and final state in a picture format and the value of the energy as a function of the time steps.

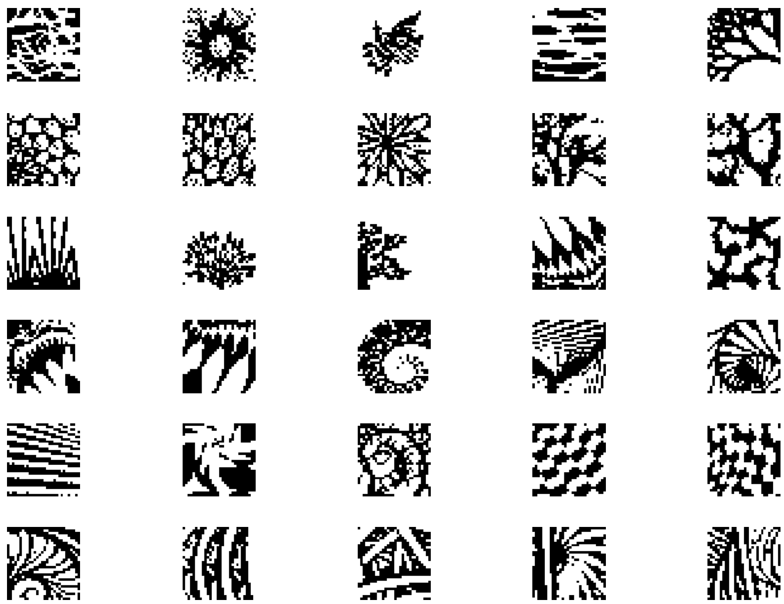

Figure 1: 30 of 900, 30×30 pictures in the bank.

**b** Repeat part a but instead of remembering one picture, add more and more pictures (increase $P$ in Eq.(6)). Start from the same (but random) initial condition. **Plot** the initial and final state in a picture format and the value of the energy as a function of the time steps for $P = 1, 5, 10, 20, 30, 40$.

**c** **In words**, are any of the final states in part b not the same as any pictures in the "bank"?

### 5.1.3 We consider a minimal Hopfield network

We assume for all $i$'s and $j \neq i$,

$$\theta_i = 0 \quad \text{and} \quad W_{ij} = \frac{1}{N}\sum_{k=1}^{P} \xi_i^k \xi_j^k. \tag{6}$$

where $\xi_i \in \{-1, +1\}$. We will investigate the memory storage and retrieval properties of the Hopfield network.

**1** **Show** that under condition Eq.(6), if $\vec{S}$ is a fixed point, then $-\vec{S}$ is a fixed point.

**2** When there is only one pattern ($P = 1$), the behavior of the Hopfield network can be fully understood. In this case, we can simply take $W_{ij} = \xi_i \xi_j$.

- **a** **Show** that the two states $\vec{S} = \vec{\xi}$ and $\vec{S} = -\vec{\xi}$ are fixed points.
- **b** Using the above results (**1**), **show** that $\vec{\xi}$ and $-\vec{\xi}$ are the only two fixed points when $P = 1$. Thus, by (**1**) , the network will finally reach the pre-stored memory up to a sign. [Hint: When $P = 1$, the energy function can be written as $E = -\frac{1}{2}(\sum_{i=1}^{N} \xi_i S_i)^2 + \frac{N}{2}$.]
- **c** **In words**, given any initial state $\vec{S}$, can you predict whether it will reach $\vec{\xi}$ state or $-\vec{\xi}$ state?