Homework: Simulating Two Coupled Oscillators

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Problem 1: Numerical simulation

Implement a numerical simulation of the system

$$\begin{cases} \dfrac{d\delta\psi}{dt} = \delta\omega + \Gamma_0 \sin(\delta\psi' - \delta\psi), \\ \dfrac{d\delta\psi'}{dt} = \delta\omega' + \Gamma_0 \sin(\delta\psi - \delta\psi'). \end{cases}$$

Procedure

1. **Initial conditions:** Choose initial phases, e.g. $\delta\psi(0) = \psi_0$, $\delta\psi'(0) = \psi'_0$. (choose initial values randomly)
2. **Simulation parameters:**
   - Set a coupling strength, e.g. $\Gamma_0 = 1$.
   - Set $\delta\omega = 0.5, \delta\omega = 0.2'$.
   - Initialize the phases at random over $[0, 2\pi)$.
3. **Numerical integration:** Use Euler method (listed at the end of the homework) to integrate from $t = 0$ to $t = T = 100$ with $dt = 0.01$.
4. **Analysis:** Plot $\Delta(t) = \delta\psi(t) - \delta\psi'(t)$.
   - In the phase-locking regime, $\Delta(t)$ will converge to a constant value.
   - Outside of this regime, $\Delta(t)$ will drift over time.
5. **Compare to theory:** If locked, check that the final Δ_∞ matches

$$\sin^{-1}\Big(\frac{\delta\omega - \delta\omega'}{2\,\Gamma_0}\Big).$$

Question

1. Set $\Gamma_0 = 0.1$ and follow the procedure outlined above. Determine whether the system exhibits phase-locking by analyzing the plot of $\Delta(t)$.

2. Repeat the question with Γ_0 from 0.05 to 0.15 with an increase of 0.01 and answer all the questions. You should see a transition from non-phase-locking to phase-locking by looking at the plots of $\Delta(t)$ for each Γ_0.

- Repeat the experiment for Γ_0 values ranging from 0.05 to 0.15 in increments of 0.01. For each Γ_0, answer all relevant questions and analyze the corresponding $\Delta(t)$ plots. You should observe a transition from a non-phase-locked state to a phase-locked state as Γ_0 increases.

Problem 2: Exploring a Small 1D Network

Extend your simulation to a 1D chain of N oscillators, $\delta\psi_1, \ldots, \delta\psi_N$, with nearest-neighbor coupling. For $1 < j < N$,

$$\frac{d\delta\psi_j}{dt} = \delta\omega_j + \Gamma_0\Big[\sin(\delta\psi_{j-1} - \delta\psi_j) + \sin(\delta\psi_{j+1} - \delta\psi_j)\Big].$$

(Adjust boundary conditions for $j = 1$ and $j = N$ as needed.)

Procedure:

1. **Implement the Simulation:**
   - Set up a numerical simulation to integrate the system of equations using Euler's below.
   - Use a chain length of $N = 20$ oscillators.
   - Assign intrinsic frequency deviations. For example, you can use $\delta\omega_j =$ np.random.uniform(-0.2, 0.2, N).
   - Initialize the phases at random over $[0, 2\pi)$.
2. **Simulation Analysis:**
   - Run the simulation for $T = 100$ and $dt = 0.01$.
   - Plot the $\delta\psi_j$ of each oscillator as a function of time.
   - Compute and plot pairwise phase differences $\delta\psi_j - \delta\psi_{j+1}$ at the end of the simulation.
   - Comment on whether the chain settles into a phase-locked state.

Question

1. Choose the $\Gamma_0 = 0.1$ and follow the procedure above. Based on the plot of pairwise distances, determine the number of clusters present.

2. Repeat the question with $\Gamma_0 = 1$.

Euler's Method

Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) of the form:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Given a step size h, the numerical approximation is computed iteratively using:

$$y_{n+1} = y_n + hf(t_n, y_n)$$