1 Two Oscillators with Exponential-decay Coupling

Consider the two weekly coupled phase oscillations,
\[ \frac{d\psi_i}{dt} = \omega + \epsilon \hat{Z}(\psi_i) \cdot \vec{P}(\psi_i, \psi_j), \ i, j = \{1, 2\} \]
the two perturb each other with a small phase shift, \( \psi_i = \delta\psi_i + \omega t \),
\[ \frac{d\delta\psi_i}{dt} = \epsilon \hat{Z}(\delta\psi_i + \omega t) \cdot \vec{P}(\delta\psi_i + \omega t, \delta\psi_j + \omega t). \]

Since \( \frac{d\delta\psi_i}{dt} \ll \omega \), we can average the perturbation over a full cycle,
\[
\frac{d\delta\psi_1}{dt} = \Gamma(\delta\psi_1, \delta\psi_2) \tag{1}
\]
\[
\frac{d\delta\psi_2}{dt} = \Gamma(\delta\psi_2, \delta\psi_1) \tag{2}
\]
where \( \Gamma(\delta\psi_i, \delta\psi_j) \) represents the averaged perturbation of oscillator \( j \) on oscillator \( i \) over a full cycle,
\[ \Gamma(\delta\psi_i, \delta\psi_j) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \hat{Z}(\delta\psi_i + \theta) \cdot \vec{P}(\delta\psi_i + \omega t, \delta\psi_j + \omega t), \tag{3} \]

note that we have replaced \( \omega t = \theta \in \{-\pi, \pi\} \). Similar to what we did in the lecture, assume that the perturbation \( \vec{P}(\cdot \cdot \cdot) \) solely depends on the phase of the other oscillator, i.e., \( \vec{P}(\delta\psi_i + \theta, \delta\psi_j + \theta) \rightarrow \vec{P}(\delta\psi_j + \theta), \)
\[
\Gamma(\delta\psi_i, \delta\psi_j) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \hat{Z}(\delta\psi_i + \theta) \cdot \vec{P}(\delta\psi_j + \theta), \tag{3} \]
\[
\Rightarrow \Gamma(\delta\psi_j - \delta\psi_i) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \hat{Z}(\theta - (\delta\psi_j - \delta\psi_i)) \cdot \vec{P}(\theta), \text{ let } \Delta_{ji} = \delta\psi_j - \delta\psi_i \]
\[
\Rightarrow \Gamma(\Delta_{ji}) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \hat{Z}(\theta - (\Delta_{ji})) \cdot \vec{P}(\theta), \tag{4} \]

where the sensitivity of phase to the perturbation is
\[ Z(\phi) = \sin(\phi), \tag{5} \]
and the perturbation \( \vec{P}(\cdot \cdot \cdot) \) is given by an exponential function
\[ \vec{P}(\phi) = \begin{cases} 
0, & \phi < 0 \\
\frac{2\pi}{\tau_m} e^{-\phi/\omega \tau}, & \phi \geq 0.
\end{cases} \tag{6} \]
Eq. (1) and (2) become
\[
\begin{align*}
\frac{d\delta\psi_1}{dt} &= \Gamma (\delta\psi_2 - \delta\psi_1) = \Gamma (\Delta_{21}), \\
\frac{d\delta\psi_2}{dt} &= \Gamma (\delta\psi_1 - \delta\psi_2) = \Gamma (\Delta_{12}).
\end{align*}
\]
Subtracting these two equations, we get the equation of motion for the phase difference between the two oscillators,
\[
\Rightarrow \frac{d}{dt}(\delta\psi_1 - \delta\psi_2) = \Gamma (\delta\psi_2 - \delta\psi_1) - \Gamma (\delta\psi_1 - \delta\psi_2),
\]
\[
\Rightarrow \frac{d(\Delta_{12})}{dt} = \Gamma (\Delta_{21}) - \Gamma (\Delta_{12}),
\]
\[
= 2\Gamma_{odd}(\Delta_{21}) = 2\Gamma_{odd}(\Delta_{12}), \quad (7)
\]
\[
= -2\Gamma_{odd}(\Delta_{12}), \quad (8)
\]
where \(\Gamma_{odd}(\cdots)\) is the odd part of the function \(\Gamma(\cdots)\).

1.1 Calculate the averaged perturbation \(\Gamma(\Delta)\) and then find the odd function \(\Gamma_{odd}(\Delta)\) (Hint: see the foot note\(^1\)).

1.2 Analyze the stability for the case where two oscillators have
(a) excitatory connections \(g_{syn} = g_{syn}^{excitatory} > 0\); (b) inhibitory connections \(g_{syn} = g_{syn}^{inhibitory} < 0\).
For each case, do the two oscillators tend to be in-phasic or anti-phasic?

2 Noise and “Balanced” Network

Assume that a neuron in a neural network receives both \(K\) excitatory and \(K\) inhibitory inputs from the presynaptic neurons, each of which is spiking \((V_j = 1)\) with probability \(m\) and silences \((V_j = 0)\) with probability \(1 - m\). The total input sending to the \(i\)-th postsynaptic neuron is
\[
\mu_i = \frac{1}{K} \sum_{j=1}^{K} W_{ij}^E V_j^E + \frac{1}{K} \sum_{j=1}^{K} W_{ij}^I V_j^I
\]
where the weights, \(W_{ij}^{E,I}\) may or may not depend on the number of connections \(K\). In this exercise, we adopt a numerical approach to explore the relation between the mean \(\langle \mu_i \rangle_T\), the variance \(\text{Var}(\mu_i)\) of total inputs versus the number of presynaptic connections \((K)\), given two scenarios, respectively:
(1) A postsynaptic neuron receives purely excitatory inputs from \(K\) presynaptic neurons \((W_{ij}^E \neq 0, W_{ij}^I = 0)\);
(2) A postsynaptic neuron receives both \(K\) excitatory and \(K\) inhibitory inputs \((W_{ij}^{E,I} \neq 0)\).

2.1 Consider a cell receiving the inputs via purely excitatory connections \((W_{ij}^E = W_0, W_{ij}^I = 0)\),
\[
\mu_i(t) = \frac{1}{K} \sum_{j=1}^{K} W_{ij}^E V_j^E(t),
\]
\[
= \frac{W_0}{K} \sum_{j=1}^{K} V_j^E(t)
\]
Apparently, for the constant connections \((W_{ij}^E = W_0)\) the input to all neurons is equal, i.e., \(\mu_i(t) = \mu(t)\) for all the cells in the network, so we just drop the subscripts \(i\).

\(^1\)The integral (Eq. 4) can be done by extending the range of integration over all time, i.e., \(\int_{-\infty}^{\infty}\). Since \(\tilde{P}(\phi) = 0\) when \(\phi < 0\), the integral in Eq. 4 becomes \(\int_{0}^{\infty}\). For more detail, please cf. Sec. 5.3.1 of Week 4 lecture notes.
To calculate the statistics about the total input $\mu$, firstly we need to generate $K$ inputs for one time step, and extend it to a sequence (i.e., multiple time steps) over the time duration $T$. Then, compute the mean $\langle \mu \rangle_T$ and variance $\text{Var}(\mu)$ from this input sequence. Here are the steps for you to follow:

**STEP 1:** Generate $K$ inputs occurring in one time step ($dt$) as an one-dimensional binary vector

$$V^E = \begin{pmatrix}
V^E_1 \\
V^E_2 \\
\vdots \\
V^E_K
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
1
\end{pmatrix}$$

*Hint:* use MATLAB command `(rand(K, 1) <= m * dt)` to generate a $K$-by-1 column vector, where $m * dt$ is the spiking probability of one time bin ($dt$). You can choose the spiking probability ($m$) between $0 < m \leq 1$.

**STEP 2:** Extend the $K$ excitatory inputs ($K$-by-1) from one-time step to a sequence over a period of time $T \gg dt$ (ex: you may set $dt = 1$ and $T = 10000$). *Hint:* how many $dt$ bins ($n$) are there in the time interval $T$? Find the number of bins ($n$) and use `(rand(K, n) <= m * dt)` to generate a $K$-by-$n$ binary matrix. Each column represents $K$ inputs at each time step, while each row is the input sequence sending from one of the presynaptic neurons.

$$V^E \{t\} = \begin{pmatrix}
1^\text{st} \\
2^\text{nd} \\
\vdots \\
K^\text{th}
\end{pmatrix} \begin{pmatrix}
V^E(0) \\
V^E(dt) \\
V^E(2dt) \\
\vdots \\
V^E(T)
\end{pmatrix}$$

Time steps from 0 to $T$

**STEP 3:** From the $K$-by-$n$ matrix, how would you get a total input $\mu$ received at each time step?

*Hint:* $(w_0/K) * \text{sum}(\text{rand}(K, n) <= m * dt, 1)$ returns the sum of each column, i.e., an 1-by-$n$ row vector. You can set any values for $w_0$.

**STEP 4:** Now you have input sequence $\mu$ (1-by-$n$ row vector), you will be able to calculate the mean $\langle \mu \rangle_T$ and the variance $\text{Var}(\mu)$. *Hint:* Try to look up some built-in MATLAB functions for computing mean and variance.

**STEP 5:** Write a for-loop that calculate the mean $\langle \mu \rangle_T$ and variance $\text{Var}(\mu)$ with different connections $K = 200, 400, 600, 800, 1000, 1500, 2000, 3500,$ and $5000$. For each $K$, you may want to run multiple trials and take the average.

**STEP 6:** Plot $\langle \mu \rangle_T$ vs $K$, and $\text{Var}(\mu)$ vs $K$. Compare each curve with zero (horizontal line $y = 0$, label it on the plot would be helpful), and briefly describe the trend for each.

### 2.2 $K$ excitatory and $K$ inhibitory inputs ($W^E_{ij} = -W^I_{ij} = W_0$),

$$\mu(t) = \frac{W_0}{K} \left( \sum_{j=1}^{K} V^E_j(t) - \sum_{j=1}^{K} V^I_j(t) \right).$$

Extend the simulation steps in **1.1**, plot $\langle \mu \rangle_T$ vs $K$ and $\text{Var}(\mu)$ vs $K$ for a case that a neuron receives both $K$ excitatory and $K$ inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), and briefly describe the trend for each.

### 2.3 $K$ excitatory and $K$ inhibitory inputs, with the weights $W^E_{ij} = -W^I_{ij} = \sqrt{K} W_0$,

$$\mu(t) = \frac{W_0}{\sqrt{K}} \left( \sum_{j=1}^{K} V^E_j(t) - \sum_{j=1}^{K} V^I_j(t) \right).$$
Extend the simulation steps in 1.1, plot $⟨μ⟩_T$ vs $K$ and Var($μ$) vs $K$ for a case that a neuron receives both $K$ excitatory and $K$ inhibitory inputs. Compare each curve with its own zero (horizontal line $y = 0$), briefly describe the trend for each, and compare them with the results you got from 2.1 and 2.2.

### 3 Stable Patterns of Oscillation in Two Time Delay Coupled Oscillators

Coupled oscillators have more complex dynamics when their interactions are time delayed. When the interactions have no time delay, there is only one stable pattern. We’ll investigate the dynamics of these oscillator pairs as the time delay varies.

The two oscillators have phases $ψ_1(t)$ and $ψ_2(t)$ and evolve in time according to the equations

\[
\frac{dψ_1(t)}{dt} = ω_1 + Γ_0 sin(ψ_1(t) − ψ_2(t − τ)) \tag{9}
\]

\[
\frac{dψ_2(t)}{dt} = ω_2 + Γ_0 sin(ψ_2(t) − ψ_1(t − τ)) \tag{10}
\]

where $τ$ is the time delay, $Γ_0$ is a maximum amplitude of the interaction, and $ω_1$ is the default frequency of oscillator 1 if there is no interaction.

A suitable guess for both solutions is that both evolve according to a constant frequency $Ω$ but are phase shifted by some constant phase $α$. That is, we guess a solution

\[
ψ_{1,2}(t) = Ωt ± \frac{α}{2} \tag{11}
\]

The $+$ and $-$ apply to $ψ_1$ and $ψ_2$ respectively.

#### 3.1 Solving for Valid Frequencies and Phase Shifts

Using the above guess for $ψ_{1,2}(t)$, show that the time delay coupled oscillators have frequencies $Ω$ that obey the following equation:

\[
0 = \frac{ω_1 + ω_2}{2} - Ω + Γ_0 tan(Ωτ) \sqrt{cos^2(Ωτ) - \frac{(ω_1 - ω_2)^2}{Γ_0^2}} \tag{12}
\]

**Hint:** Try taking sums and differences of equations 9 and 10 with the solution (11) plugged in. You may require the identities $sin(α + β) = sin(α)cos(β) + cos(α)sin(β)$ and $sin^2(x) = 1 - cos^2(x)$ at some point during the calculation.

#### 3.2 Plotting the Oscillation Frequencies - Time Delay Nonzero

Equation 12 is a transcendental equation which allows us to find the frequency $Ω$ (assuming that we know $ω_1$, $ω_2$, $Γ_0$, and $τ$). Find $Ω$ by plotting the curve given by the right hand side of equation 12 for a couple different combinations of $τ$ ≠ 0 and $Γ_0$ ≠ 0. List the resulting allowed $Ω$. The values of $Ω$ at which the curve intersects 0 are the allowed values of $Ω$. Assume $ω_1 = 1.2$ and $ω_2 = 0.8$.

#### 3.3 Finding the Oscillation Frequencies - Time Delay Zero

Briefly provide an expression that allows one to determine the allowed $Ω$ values when $τ = 0$. How many allowed $Ω$ are there? It’s not necessary to provide a plot in the same manner as 3.2, but it can be helpful.

#### 3.4 Stability of Solutions

We can perturb the solution $ψ_{1,2}(t) = Ωt ± \frac{α}{2}$ by adding a term that grows or decays exponentially with time (determined by parameter $λ$). The perturbed solution in full is $ψ_{1,2}(t) = Ωt ± \frac{α}{2} + e^{λt}$, where parameter $ε$ is some small displacement amplitude. When linearizing around the equilibrium point, it is useful to treat $ψ_1(t)$, $ψ_1(t − τ)$, $ψ_2(t)$, and $ψ_2(t − τ)$ as separate variables. Recall that the linearization of the time derivative equations around a fixed point yields a "characteristic equation" that can be used to solve for the eigenvalues $λ$, which in turn tells us the dynamics near the equilibrium point. Here we use an equilibrium solution $ψ_{1,2}(t) = Ωt ± \frac{α}{2}$ instead of a point. Derive the characteristic equation for this time delay coupled oscillator system. The equation is

\[
0 = det \begin{pmatrix}
-λ + Γ_0 cos(Ωτ + α) \\
-Γ_0 e^{-λτ} cos(Ωτ + α) \\
-λ + Γ_0 cos(Ωτ - α) \\
-Γ_0 e^{-λτ} cos(Ωτ - α)
\end{pmatrix}
\]

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2 All problem 3 material based on Schuster and Wagner paper Mutual Entrainment of Two Limit Cycle Oscillators with Time Delayed Coupling (1989). See PHYS 178/278 Winter 2021 Lecture 14 notes for further information. Both resources can help with this problem.