Phys 178: HW4

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March 2024

Due midnight on March 18th. Please justify all of your answers, make intermediate plots and submit to Gradescope. Make sure you select the correct question for each part of your submission, otherwise it will be graded as missing. If you have any questions, please email Ghita Guessous (gguessou@ucsd.edu).

1 Two Oscillators with Exponential-decay Coupling

Consider the two weakly coupled phase oscillations

$$\frac{d\psi_i}{dt} = \omega + \epsilon \vec{Z}(\psi_i) \cdot \vec{P}(\psi_i, \psi_j), \quad i, j = \{1, 2\}, \tag{1}$$

where the two perturb each other with a small phase shift $\psi_i = \delta\psi_i + \omega t$. The dynamics of the phase shift $\delta\psi_i$ are given by

$$\frac{d\delta\psi_i}{dt} = \epsilon \vec{Z}(\delta\psi_i + \omega t) \cdot \vec{P}(\delta\psi_i + \omega t, \delta\psi_j + \omega t). \tag{2}$$

Since $\frac{d\delta\psi_{ij}}{dt} \ll \omega$, we can average the perturbation over a full cycle

$$\frac{d\delta\psi_1}{dt} = \Gamma(\delta\psi_1, \delta\psi_2), \tag{3}$$

$$\frac{d\delta\psi_2}{dt} = \Gamma(\delta\psi_2, \delta\psi_1) \tag{4}$$

where $\Gamma(\delta\psi_i, \delta\psi_j)$ represents the averaged perturbation of oscillator j on oscillator i over a full cycle

$$\Gamma(\delta\psi_i, \delta\psi_j) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta Z(\delta\psi_i + \theta) \cdot P(\delta\psi_i + \theta, \delta\psi_j + \theta). \tag{5}$$

note that we have replaced $\omega t = \theta \in \{-\pi, \pi\}$.

Similar to what we did in the lecture assume that the perturbation $\vec{P}$ solely depends on the phase of the other oscillator i.e. $\vec{P}(\delta\psi_i + \theta, \delta\psi_j + \theta) \to P(\delta\psi_j + \theta)$,

$$\Gamma(\delta\psi_i, \delta\psi_j) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \vec{Z}(\delta\psi_i + \theta) \cdot \vec{P}(\delta\psi_j + \theta), \text{change of variable } \theta \to \theta - \delta\psi_j \tag{6}$$

$$\Rightarrow \Gamma(\delta\psi_j - \delta\psi_i) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \vec{Z}(\theta - (\delta\psi_j - \delta\psi_i)) \cdot \vec{P}(\theta) \text{let } \Delta_{ji} = \delta\psi_j - \delta\psi_i \tag{7}$$

$$\Rightarrow \Gamma(\Delta_{ji}) = \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} d\theta \vec{Z}(\theta - (\Delta_{ji})) \cdot \vec{P}(\theta) \tag{8}$$

where the sensitivity of phase to the perturbation is given by:

$$Z(\phi) = \sin(\phi), \tag{9}$$

and the perturbation $\vec{P}$ is given by an exponential function:

$$\vec{P}(\phi) = \begin{cases} 0 & \phi < 0 \\ \frac{g_{\text{syn}}}{c_m} e^{-\phi/\omega\tau} & \phi \geq 0. \end{cases} \tag{10}$$

Finally, the equations for the phase shifts become

$$\frac{d\delta\psi_1}{dt} = \Gamma(\delta\psi_2 - \delta\psi_1) = \Gamma(\Delta_{21}), \tag{11}$$
$$\frac{d\delta\psi_2}{dt} = \Gamma(\delta\psi_1 - \delta\psi_2) = \Gamma(\Delta_{12}). \tag{12}$$

Subtracting these two equations, we get the equation of motion for the phase difference between the two oscillators

$$\Rightarrow \frac{d(\delta\psi_1 - \delta\psi_2)}{dt} = \Gamma(\delta\psi_2 - \delta\psi_1) - \Gamma(\delta\psi_1 - \delta\psi_2) \tag{13}$$
$$\Rightarrow \frac{d(\Delta_{12})}{dt} = \Gamma(\Delta_{21}) - \Gamma(\Delta_{12}) \tag{14}$$

$$= 2\Gamma_{\text{odd}}(\Delta_{21}) = 2\Gamma_{\text{odd}}(-\Delta_{12}) \tag{15}$$
$$= -2\Gamma_{\text{odd}}(\Delta_{12}) \tag{16}$$

where $\Gamma_{\text{odd}}(\cdot)$ is the odd part of the function $\Gamma(\cdot)$.

1.1 Calculate the averaged perturbation $\Gamma(\Delta)$ and then find the odd function $\Gamma_{odd}(\Delta)$ (Hint: see the footnote [1]).

1.2 Analyse the stability for the case where the two oscillators have:

- **a** excitatory connections $g_{syn} = g_{syn}^{excitatory} > 0$
- **b** inhibitory connections $g_{syn} = g_{syn}^{inhibitory} < 0$

For each case, do the two oscillators tend to be phasic or anti-phasic?

[1] The integral (Eq. 8) can be done by extending the range of integration over all time i.e. $\int_{-\infty}^{\infty}$. Since $\vec{P}(\phi) = 0$ when $\phi < 0$ the integral in Eq. 8 becomes $\int_0^{\infty}$. For more detail please cf. Sec. 5.3.1 of Week 4 lecture notes.