8 Storage Capacity in Associative Networks for (Near) Perfect Recall

We consider a Hopfield network with the standard Hebb-like learning rule and ask how many memories we can imbed in a network of N neurons with the constraint that we will accept at most one bit (one neuron's output in only one memory state) of error

$$\begin{aligned} Input \quad &= \sum_{j \neq i}^{N} W_{ij} S_j \\ &= \frac{1}{N} \sum_{\mu=1}^{p} \sum_{j \neq i}^{N} \zeta_i^\mu \zeta_j^\mu S_j \end{aligned} \tag{8.201}$$

where p is the number of stored memories, N is the number of neurons and

$$W_{ij} \equiv \frac{1}{N} \sum_{\mu=1}^{p} \zeta_i^\mu \zeta_j^\mu \tag{8.202}$$

is the synaptic weight matrix given by the Hebb rule.

Now, check stability of stored state. Make $S_j = \zeta_j^1$, one of the stored memory states, so that

$$\begin{aligned} Input \quad &= \frac{1}{N} \sum_{\mu=1}^{p} \sum_{j \neq i}^{N} \zeta_p^\mu \zeta_j^\mu \zeta_j^1 \\ &= \frac{1}{N}\, \zeta_j^1 \sum_{j \neq i}^{N} \zeta_i^1 \zeta_j^1 + \frac{1}{N} \sum_{\mu \neq 1}^{p} \zeta_i^\mu \sum_{j \neq i}^{N} \zeta_j^\mu \zeta_j^1 \end{aligned} \tag{8.203}$$

On average, the second term is zero, so that

$$\text{Average of Input} = \left(\frac{N-1}{N} \right) \zeta_i^1 \;\simeq\; \zeta_i^1 \tag{8.204}$$

What is the variance? The second term, summed over random vectors with zero mean, consists of the sum of $(p-1)$ inner products of vectors with $(N-1)$ terms. Each term is +1 or -1, i.e., binomially distributed, so that

$$\text{Variance of Input} = \left(\frac{1}{N} \right)^2 \cdot (p-1) \cdot (N-1) \simeq \frac{p}{N} \tag{8.205}$$

This results in a fluctuation to the input with a standard deviation, σ, of

$$\sigma = \pm\sqrt{P/N} \tag{8.206}$$

Noise hurts only if the magnitude exceeds 1. The noise becomes Gaussian for large p and N, which is the limit of interest, Thus the probability of an error in the recall of all stored states is

$$\begin{aligned}
P_{error} &= \frac{1}{\sqrt{2\pi}\ \sigma}\ \left[\ \int_{\infty}^{-1}\ e^{-x^2/2\sigma^2}\ dx\ +\int_{+1}^{\infty}\ e^{-x^2/2\sigma^2}\ dx\ \right] && (8.207)\\
&= \frac{\sqrt{2}}{\sqrt{\pi}\ \sigma}\ \int_{+1}^{\infty}\ e^{-x^2/2\sigma^2}\ dx && (8.208)\\
&= \frac{2}{\sqrt{\pi}}\ \int_{\sqrt{\frac{N}{2p}}}^{\infty}\ e^{-x^2}\ dx && (8.209)\\
&\equiv \mathrm{erfc}\left(\sqrt{\frac{N}{2p}}\right)
\end{aligned}$$

where efrc(x) is the complementary error function and we again note that the average of the error term is zero. We recall that for $\frac{N}{2p} << 1$, the complementary error fucntion may be approximated by an asymptotic form, so that

$$P_{error} \simeq \frac{1}{\sqrt{\pi}}\frac{2p}{N}e^{-N/2p} \tag{8.210}$$

We have a nice and closed expression in a relevant limit!

Now $N \cdot p$ is total number of "bits" in the network. Suppose only less than one bit can be in error. Then

$$(1 - P_{error})^{Np} \geq 1 - \frac{1}{Np} \tag{8.211}$$

But Np is large and P_{error} will be small by construction, so $1 - Np \times P_{error} \geq 1 - \frac{1}{Np}$ and thus

$$P_{error} < \frac{1}{(Np)^2} \tag{8.212}$$

From the expansion of the gaussian error:

$$log\,[P_{error}] \simeq -\frac{1}{2}\ log\ \pi - \frac{N}{2p} - log\ \frac{N}{2p} \tag{8.213}$$

From the constraint on the desired error:

$$log\left[P_{error}\right] < -2\ log(Np) \tag{8.214}$$

Thus

$$-\frac{1}{2}\ log\ \pi - \frac{N}{2p} - log\ \frac{N}{2p}\ < -2\ log\ (Np) \tag{8.215}$$

We now let $N \to \infty$ with N/p constant. To leading order, we have:

$$\frac{N}{2p} > 2\ log\ (Np) \tag{8.216}$$

To go further and solve for p in terms of N, we assume that

$$p = \alpha(N)N \tag{8.217}$$

then

$$\frac{N}{2\alpha N}\ >\ 2\ log\ (N\alpha N) \tag{8.218}$$

$$1 > 8\alpha\ log\ N \tag{8.219}$$

and therefore

$$p\ <\ \frac{1}{8}\ \frac{N}{log\ N} \tag{8.220}$$

Note that p has a similar scaling for the choice of a fixed error rate.

Thus we see that an associate memory based on a recurrent Hopfield network stores a number of memories that scales more weakly than the number of neurons if one cannot tolerate any errors upon recall. If a finite number of errors can be tolerated, a statistical mechanical deviation shows $p < 0.14N$.