Physics 171/271 -David Kleinfeld - Fall 2005 (revised Winter 2011)

1 Derivation of Rate Equations from Single-Cell Conductance (Hodgkin-Huxley-like) Equations

We consider a network of many neurons, each of which obeys a set of conductance-based equations for the transmembrane potential that leads to subthreshold and spike activity. Our goal is to derive an expression for the firing rate of the neuron in terms of the synaptic input to the cell. The equations are solved in a self-consistent manner in the sense that the output of each neuron contributes to the synaptic input of every other neuron.

We are motivated to this proof as a means of connecting single-cell equations, which are complicated, to simplified network equation, which are simple in that each cell is specified solely by its firing rate. Our goal is to understand the weights W_{ij} in terms of cellular properties. We follow the derivation laid down by Sompolinsky, which holds for averaging over inputs that arrive with a Poisson (exponential) distribution for a large networks. A similar derivation, which holds for averaging over a large number of spikes by a single synaptic input, was given by Ermentrout. In each case the critical issue is that the network dynamics are asynchronous.

Kirchoff's Law for One Neuron

$$C\,\frac{dV(t)}{dt} + g_L\left[V(t)\ - E_L\right] + I^{active}(t) = I^{applied}(t) \tag{1.1}$$

where E_L = reversal potential of leak, I^{active} (t) = all time-dependent active currents, such as the Na^+, K^+, and Ca^{2+} active currents, and $I^{applied}$ (t) = all time-dependent applied currents. The applied current contains terms from external inputs as well as lateral, or network interactions. Thus,

$$C\,\frac{dV(t)}{dt} = g_L\left[E_L - V(t)\right] - I^{active}\,(t) + I^{applied}\,(t) \tag{1.2}$$

where g_L is the leakage conductance of the membrane and E_L is the leakage reversal potential, nominally the resting potential when all synaptic inputs are turned off.

Let

$$\Lambda = f\left[I^{applied}\ -\ I^{offset}\right] \tag{1.3}$$

where = $f\left(I^{applied}\ -\ I^{offset}\right)$ = firing rate, f is the functional form of the firing input-output curve. This relation gains possible time dependence through the time dependence of it's arguments, and for semi-linear relations such as $\Lambda = f\,[I]_+$ has units of inverse Coulombs. We incorporate the leak conductance through

$$I^{offset} = I_c^{offset} \; + E_c \; g_L \tag{1.4}$$

where the constant E_c scales the dependence of firing threshold on the leak conductance. Thus

$$\Lambda = f\left[I^{applied} - I_c^{offset} \; - E_c \; g_L\right]. \tag{1.5}$$

This formalism assumes that changes in conductance shifts the threshold level of the $f-I$ curve, but does not effect the slope of the curve. This is approximately true, at least based on the experiments of Reyes and Abbott (Neuron). It corresponds to a shifting threshold for the input-output curve rather than a change in the gain or slope of the curve.

Network Equations

$$C\frac{dV_i(t)}{dt} = g_L\left[E_L - V_i(t)\right] + I_i^{active}(t) + I^{external}(t) + I^{network}(t) \tag{1.6}$$

where we have sit the applied inputs into two terms: inputs from the "external world" come in through $I^{external}(t)$ and inputs from other cells in the network (lateral or feedback connections) come in through $I^{network}(t)$. The $g_L\left[E_L - V_i(t)\right]$ term will term to be pivotal as we will augment the leakage conductance with synaptic conductances.

We consider the form of $I_i^{external}(t)$, the input to the i-th neuron, first, i.e.,

$$I_i^{external} = g_i^{in}(t)\left[E_{in} - V_i(t)\right] \tag{1.7}$$

The conductance weights the external input and is described by a first order equation

$$\tau_{in}\,\frac{dg_i^{in}}{dt} + g_i^{in} = G_{in}\;\tau_{in}\;\lambda_{in}(t) \quad ; t \geq 0 \tag{1.8}$$

where the input $\lambda_{in}(t)$ is taken to be a Poisson process with mean rate Λ_i^{in}.

The formal solution for the state (steady state of inhomogeneous response) is given by

$$g_i^{in}(t) = G_{in}\;\int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{in}}\;\lambda^{in}(t) \tag{1.9}$$

so that the average over a large number of inputs, either separate presynaptic inputs or many post-synaptic potential from one input by a very slow synapse, is

$$\begin{aligned} \left\langle g_i^{in}(t)\right\rangle \quad &= G_{in} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{in}} \left\langle \lambda^{in}(t)\right\rangle \\ &= G_{in}\ \Lambda^{in} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{in}} \\ &\simeq G_{in}\ \Lambda^{in} \int_{-\infty}^{\infty} dt' e^{-(t-t')/\tau_{in}} \\ &= G_{in}\ \Lambda^{in} \tau_{in} \end{aligned} \tag{1.10}$$

where the product $\Lambda^{in}\tau_{in}$ is just the number of post-synaptic inputs (or pre-synaptic spikes from all neighbors, since we do not include synaptic depression and other time-dependent synaptic effects) that occur in the time-constant of the post-synaptic cell.

Look at ratio of mean to standard deviation, i.e., $\frac{1}{\sqrt{\tau^{in}\Lambda^{in}}}$. This fraction decreases as either the time-constant goes to infinity or the frequency goes to infinity.

We now have

$$I_i^{external} \rightarrow G_{in}\ \tau_{in}\Lambda^{in}\left[E_{in} - V_i(t)\right] \tag{1.11}$$

We turn to the current that results from synaptic inputs, i.e.,

$$I_i^{netwrk} \quad \equiv \quad \sum_{j=1}^{n} g_{ij}(t)\left[E_j - V_i(t)\right] \tag{1.12}$$

where $g_{ij}(t)$ is the post-synaptic conductance triggered by pre-synaptic spike

$$\tau_{ij}\ \frac{dg_{ij}}{dt} +\ g_{ij} = \tau_{ij}\ G_{ij} \sum_{t_j}^{all\ spikes} \delta\ (t - t_j)\ ;\ t \geq 0 \tag{1.13}$$

where the summation is over the spikes in pre-synaptic neuron "j" and we recall that the units of $\delta(t)$ are 1/s.

We replace the spatial summation in synaptic input by the ensemble average, as in the case of the external input. This holds for Poisson firing rates among the neurons in the network. Thus

$$g_{ij}(t) = G_{ij} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{ij}} \sum_{t_j}^{all\ spikes} \delta\ (t - t_j) \tag{1.14}$$

so that the average over a large number of inputs (either separate presynaptic inputs or many post-synaptic potential from one input by a very slow synapse) is

$$\begin{aligned}\langle g_{ij}(t)\rangle &= G_{ij}\int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{ij}} \left\langle \sum_{t_j}^{all\ spikes} \delta\left(t-t_j\right)\right\rangle \\ &= G_{ij}\ \Lambda_j \int_{-\infty}^{t} dt' e^{-(t-t')/\tau_{ij}} \\ &\simeq G_{ij}\ \Lambda_j \int_{-\infty}^{\infty} dt' e^{-(t-t')/\tau_{ij}} \\ &= G_{ij}\ \Lambda_j \tau_{ij}\end{aligned} \tag{1.15}$$

where the integral is just $\tau_{ij}\Lambda_j$ = mean number of spikes. The network contribution to the current to the cell become

$$I_i^{network} \to \sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ \Lambda_j \left[E_j - V_i(t)\right] \tag{1.16}$$

The sum of the external and network currents is thus

$$\begin{aligned}I_i^{applied} &= I_i^{external} + I_i^{network} \\ &= G^{in}\ \tau^{in}\ \Lambda_i^{in}\left[E_{in} - V_i(t)\right] + \sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ \Lambda_j\left[E_j - V_i(t)\right]\end{aligned} \tag{1.17}$$

where N is the number of neurons in the network. This expression has constant terms and voltage dependent terms. Let's put all the voltage terms with respect to E_L, so that the input gains a term that appears in form similar to that of g_L. That is. we add and subtract terms to form

$$\begin{aligned}I_i^{applied} &= G_{in}\tau_{in}\Lambda_i^{in}\ (E_{in} - E_L) + \sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ \Lambda_j\ (E_j - E_L) \\ &+ \left(G_{in}\ \tau_{in}\ \Lambda_i^{in} + \sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ \Lambda_j\right)\left[E_L - V_i(t)\right]\end{aligned} \tag{1.18}$$

The first two terms to the right of the equality are constant, i.e., independent of $V(t)$. We consider this an an effective applied current, denoted $I_i^{applied_effective}$

$$I_i^{applied-effective} \equiv G_{in}\tau_{in}\Lambda_i^{in}\ (E_{in} - E_L) + \sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ \Lambda_j\ (E_j - E_L) \tag{1.19}$$

The important thing is that the external and network terms have a voltage dependence that is proportional to the difference between the synaptic reversal potential and the leakage reversal potential, i.e., the resting potential.

The second term appears as a leakage current, in which the synaptic input adds to this leakage. We define an effective synaptic conductance, $g_i^{synapse-shunt}$ that adds to the leakage conductance g_L, i.e,

$$g_i^{synapse-shunt} \equiv G_{in}\ \tau_{in}\ \Lambda_i^{in} + \sum_{j=1}^{N}\ G_{ij}\ \tau_{ij}\ \Lambda_j \tag{1.20}$$

The results of our efforts is that we can write Kirchhoff's law for one neuron, rather than the network, with effective parameters. In particular, $(g_L + g_i^{synapse-shunt})$ is the effective leak current.

$$C\ \frac{dV_i(t)}{dt} = -(g_L + g_i^{synapse-shunt})\,[E_L - V_i(t)] - I_i^{active}(t) + I_i^{applied} \tag{1.21}$$

The interactions between neurons enter this equation though through the synaptic inputs and their effect on $g_i^{synapse-shunt}$. We find this term from the modified firing rate as

$$\begin{aligned}
\Lambda_i \quad &= f\left[I_i^{applied} - I_c^{offset} - E_c\ g_L - E_c\ g_i^{synapse-shunt}\right] \\
&= f\left[G_{in}\tau_{in}\Lambda_i^{in}(E^{in} - E_L) + \sum_{j=1}^{N} G_{ij}\tau_{ij}\Lambda_j(E_j - E_L) - I_c^{offset} - g_L E_c - G_{in}\tau_{in}\Lambda_i^{in}E_c - \sum_{j=1}^{N} G_{ij}\tau_{ij}\ \Lambda_j E_c\right] \\
&= f\left[\sum_{j=1}^{N} G_{ij}\ \tau_{ij}\ (E_j - E_L - E_c)\ \Lambda_j\ +\ G_{in}\ \tau_{in}\ (E_{in} - E_L - V_c)\ \Lambda_i^{in}\ -\ \left(I_c^{offset} + g_L E_c\right)\right]
\end{aligned} \tag*{(1.22}$$

It is worth noting that the equations were derived under the assumption that spiking is a Poisson process. Clearly a single cell model can be deterministic. This one must add thermal noise or parameter variation to randomize the actual time of spiking about the desired norm.

We can now identify terms in the conductance equations that correspond to terms in our network equations. In particular,

Synapses

$$W_{ij} \equiv G_{ij}\tau_{ij}(E_j - E_L - E_c) \tag{1.23}$$

is the synaptic efficiency, or connection strength, between neurons in the network.

External Drive

$$W^{in} \equiv G_{in}\tau_{in}(E_{in} - E_L - E_c) \tag{1.24}$$

is the synaptic efficiency, or strength, for an external input to the neuron.

Threshold

$$\theta \equiv I_c^{offset} + g_L E_c \tag{1.25}$$

is the threshold denoted θ. Thus the equation(s) for the firing rate of the cell becomes

Rate Equation

$$\Lambda_i = f\left[\sum_{j=1}^{N} W_{ij}\lambda_j + W^{in}\Lambda_i^{in} - \theta\right] \tag{1.26}$$

We now have derived the rate equations, under the assumption that the threshold of the firing rate versus current curve is shifted by the changes in synaptic conductance and that the cell receives multiple inputs (PSPs) during each integration period (nominally τ_{ij}).

It is not surprising that the synaptic efficiency depend on the conductances, i.e., $W_{ij} \propto G_{ij}$. It is surprising that the synaptic efficiency depends on the combination $E_j - E_L - E_c$; the synaptic reversal potential must be taken relative to $E_L + E_c$.

<u>Main Result</u>

A final point is that we need a differential equation for the rates λ_i, or for the discrete case a difference equation, so that the rates evolve over time. This is equivalent to letting the input to the neuron evolve with the time-constant of the neuron. Thus for the continuous case, using our previous notation μ_i as the input to the cell, we have

$$\tau_i \frac{du_i(t)}{dt} + u_i = \sum_{j=1}^{N} W_{ij}\lambda_j(t) + W^{in}\lambda_i(t)^{in} - \theta \tag{1.27}$$

and, for example, take f to be semi-linear:

$$\Lambda_i = f\left[\mu_i\right]_+ \tag{1.28}$$

While for the discrete case, we have simply

$$\Lambda_i(t+1) \leftarrow f\left[\sum_{j=1}^{N} W_{ij}\Lambda_j(t) + W^{in}\Lambda_i^{in}(t) - \theta\right]_+ \tag{1.29}$$

In terms of the previous symmetric and normalized notation, we have $S_i = (2\lambda_i - \lambda^{max})/2\lambda^{max}$, which ranges between -1 and +1. For the special case of binary neurons, i.e., $S_i = \pm 1$, which corresponds to $\lambda \to \infty$ with a saturating firing rate, λ^{max}, we have

$$S_i(t+1) \leftarrow \left[\sum_{j=1}^{N} W_{ij}S_j(t) + W^{in}S_i^{in}(t) - \theta\right]_+ \tag{1.30}$$

Response of Homogeneous Population of Neurons

We consider a homogeneous population, the simplest network, to gain insight into behavior of the network

$$W_{ij} \rightarrow W \text{ and } \Lambda_i^{in} \rightarrow \Lambda^{in}$$

therefore

$$\Lambda_i(t+1) = f\left[W \sum_{j=1}^{N} \Lambda_j(t) + W^{in}\Lambda^{in} - \theta\right]_+ \tag{1.31}$$

and homogeneity across all neurons gives a steady-state ($t \rightarrow \infty$) result of

$$\Lambda = f\left[NW\Lambda + W^{in}\Lambda^{in} - \theta\right]_+ \tag{1.32}$$

where the rate is the same for all neurons ($\Lambda_i \rightarrow \Lambda$). Thus

if $NW\Lambda < W^{in}\Lambda^{in} - \theta$ then $\quad \Lambda = 0$

if $NW\Lambda > W^{in}\Lambda^{in} - \theta$ then $\quad \Lambda \neq 0$

For the nonzero case:

$$\Lambda = fNW\Lambda + f(W^{in}\Lambda^{in} - \theta) \tag{1.33}$$

where we us the capital Λ since the output is piecewise linear,or

$$\Lambda = \frac{W^{in}\Lambda^{in} - \theta}{1 - fNW} \tag{1.34}$$

We see that the average firing rate, Λ, monotonically increases as the number of neurons, N, increases, until it reaches a singularity at $N = 1/(Wf)$. In reality, saturation occurs first, which can be included by adding, for example, a term proportional to -$[I - I_c]_+^2$ to the gain $(f - I)$ function, which still allows us to get an analytical result.