Physics 173 / BGGN 266 Primer on Filters and Modularity

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STEADY STATE ANALYSIS

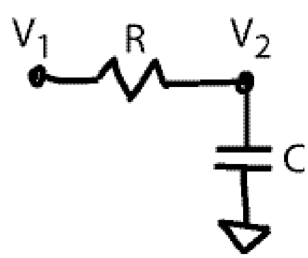
Our circuit analysis is much simplified if we make use of the concept of complex impedance, which is appropriate for steady-state signals in terms of the spectral, or frequency, content.

For a capacitor we know that $I = C\frac{dV}{dt}$. For V of the form $V(t) = \int_{-\infty}^{-\infty} d\omega\ V(\omega)\ e^{-i\omega t}$, the relation between I(ω) and V(ω) is $I(\omega) = i\omega C\ V(\omega)$ and we identify $Z_C(\omega) = \frac{1}{i\omega C}$ as the impedance - an effective resistance that affects the phase as well as the amplitude of the signal in a frequency dependent manner - of the capacitor.

Similarly, for an inductor we know that $I = \frac{1}{L}\int_{-\infty}^{t} d\tau\ V$ and we identify $Z_L(\omega) = i\omega L$ as the impedance of the inductor.

The approach of complex impedances allows us to solve algebraic equations in the frequency domain (again, for steady state) rather than differential equations (convolution integrals) in the time domain.

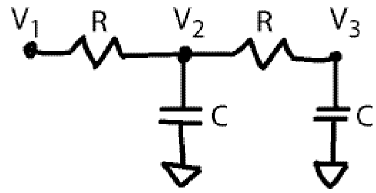
As an example, consider a RC low-pass filter, i.e.,



Kirchoff's Law gives $\frac{V_2 - V_1}{R} + V_2(i\omega C) = 0$ or $V_2 = \frac{1}{1+i\omega RC}V_1$. A little rearrangement leads to $V_2 = \frac{1-i\omega RC}{1+(\omega RC)^2}V_1$, or in vector (phasor to the "*j*" crowd) form, $V_2 = \frac{e^{-i\tan^{-1}(\omega RC)}}{\sqrt{1+(\omega RC)^2}}V_1$.

CASE OF A TWO-POLE LOW-PASS FILTER WITHOUT BUFFERING

Consider a circuit that has a sequence of two RC filters with node voltages $V_1(\omega)$, $V_2(\omega)$, and $V_3(\omega)$, i.e.,



Kirchoff's Law gives

$$\frac{V_2 - V_1}{R} + V_2(i\omega C) + \frac{V_2 - V_3}{R} = 0$$

and

$$\frac{V_3 - V_2}{R} + V_3(i\omega C) = 0\,.$$

Thus

$$V_2(2 + i\omega RC) - V_3 = V_1$$

and

$$V_2 = V_3(1 + i\omega RC)\,.$$

Combining gives:

$$V_3\left[(1 + i\omega C)(2 + i\omega RC) - 1\right] = V_1$$

so that

$$V_3 = \frac{1}{1-(\omega RC)^2 + i3\omega RC}V_1 \quad = \quad \frac{1-(\omega RC)^2 - i3\omega RC}{\left[1-(\omega RC)^2\right]^2 + (3\omega RC)^2}V_1 \quad = \quad \frac{e^{-i\tan^{-1}\left(\frac{3\omega RC}{1-(\omega RC)^2}\right)}}{\sqrt{\left[1-(\omega RC)^2\right]^2 + (3\omega RC)^2}}V_1$$

or

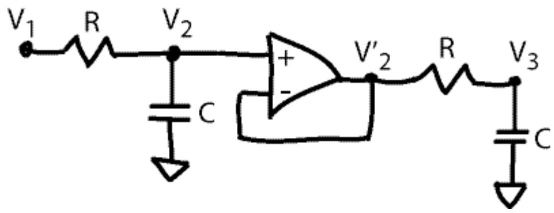
$$V_3 = \frac{e^{-i\tan^{-1}\left(\frac{3\omega RC}{1-(\omega RC)^2}\right)}}{\sqrt{1+7(\omega RC)^2 + (\omega RC)^4}}V_1 \quad = \quad \frac{e^{-i\tan^{-1}\left(\frac{3\omega RC}{1-(\omega RC)^2}\right)}}{\sqrt{\left[6.854+(\omega RC)^2\right]\left[0.146+(\omega RC)^2\right]}}V_1$$

from which we see that the break frequencies have considerably moved from $(\omega RC) = 1$. By the way, the asymptotic behavior is:

$$V_3 \xrightarrow{\omega\to\infty} \frac{e^{-i\pi}}{(\omega RC)^2}V_1 \quad = \quad -\frac{1}{(\omega RC)^2}V_1$$

CASE OF A TWO-POLE LOW-PASS FILTER WITH BUFFERING

We now consider the rather different situation that occurs when an OpAmp is used to isolate the two stages of the RC filter i.e.,



The node voltages are $V_1(\omega)$ and $V_2(\omega)$ for the first stage and $V_2'(w)$ and $V_3(w)$ for the second stage, where from our model for the OpAmp,

$$V_2'(\omega) = \frac{A(\omega)}{1+A(\omega)} V_2(\omega) \xrightarrow[\text{Frequencies}]{\text{Low}} V_2(\omega)$$

Kirchoff's Law gives

$$\frac{V_2 - V_1}{R} + V_2(i\omega C) = 0$$

or

$$V_2(1 + i\omega RC) = V_1$$

and

$$\frac{V_3 - V_2}{R} + V_3(i\omega C) = 0$$

or

$$V_3(1 + i\omega RC) = V_2$$

so that

$$V_3 = \frac{1}{(1+i\omega RC)} V_2 \;=\; \frac{1}{(1+i\omega RC)^2} V_1 \;=\; \frac{1}{1-(\omega RC)^2 + i2\omega RC} V_1 \;=\; \frac{1-(\omega RC)^2 - i2\omega RC}{\left[1-(\omega RC)^2\right]^2 + (2\omega RC)^2} V_1$$

or

$$V_3 = \frac{e^{-i\tan^{-1}\left(\frac{2\omega RC}{1-(\omega RC)^2}\right)}}{\sqrt{\left[1-(\omega RC)^2\right]^2 + (2\omega RC)^2}} V_1 \;=\; \frac{e^{-i\tan^{-1}\left(\frac{2\omega RC}{1-(\omega RC)^2}\right)}}{1+(\omega RC)^2} V_1$$

which shows that the break frequencies are both at $\omega RC = 1$. The asymptotic behavior is, as above:

$$V_3 \xrightarrow{\omega\to\infty} \frac{e^{-i\pi}}{(\omega RC)^2} V_1 \;=\; -\frac{1}{(\omega RC)^2} V_1 .$$

An important issue is to realize that the relationship between V_3 and V_1 is just the product of the response of two single pole filter stages. If we back-up a bit, and consider a single stage, Kirchoff's Law gives:

$$\frac{V_2 - V_1}{R} + V_2(i\omega C) = 0$$

so that

$$V_2 = \frac{1}{(1+i\omega RC)} V_1 \quad = \quad \frac{1 - i\omega RC}{1+(\omega RC)^2} V_1 \quad = \quad \frac{e^{-i\ \tan^{-1}(2\omega RC)}}{\sqrt{1+(\omega RC)^2}} V_1$$

The relation between V3 and V2 in the original problem (Fig. 2) is given by the square of this, i.e.,

$$V_3 = \left[\frac{e^{-i\ \tan^{-1}(2\omega RC)}}{\sqrt{1+(\omega RC)^2}}\right]^2 V_1 \quad = \quad \frac{e^{-i\ 2\tan^{-1}(\omega RC)}}{1+(\omega RC)^2} V_1$$

Recall that

$$\begin{aligned} e^{-i\ 2\tan^{-1}(\omega RC)} \quad &= \quad \cos\left[2\tan^{-1}(\omega RC)\right] - i\sin\left[2\tan^{-1}(\omega RC)\right] \\ &= \quad \cos^2\left[\tan^{-1}(\omega RC)\right] - \sin^2\left[\tan^{-1}(\omega RC)\right] - i2\sin\left[\tan^{-1}(\omega RC)\right]\cos\left[\tan^{-1}(\omega RC)\right] \\ &= \quad \frac{1}{1+(\omega RC)^2} - \frac{(\omega RC)^2}{1+(\omega RC)^2} - i2\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}\frac{1}{\sqrt{1+(\omega RC)^2}} \quad = \quad \frac{1-(\omega RC)^2 - i2\omega RC}{1+(\omega RC)^2} \\ &= \quad e^{-i\ \tan^{-1}\left(\frac{2\omega RC}{1-(\omega RC)^2}\right)} \end{aligned}$$

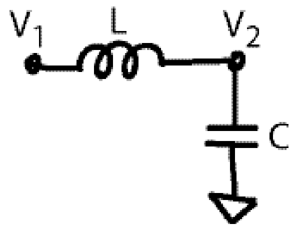
and we recover the previous result of

$$V_3 = \frac{e^{-i\ \tan^{-1}\left(\frac{2\omega RC}{1-(\omega RC)^2}\right)}}{1+(\omega RC)^2} V_1 .$$

In general, as we employ N RC stages (poles) that are separated by OpAmp buffers, the amplitude falls off as $\left[1+(\omega RC)^2\right]^{-N/2} \xrightarrow{\omega\to\infty} (\omega RC)^{-N}$. The phase is a complicated, albeit smooth function that varies from 0 at $\omega = 0$ to $N\pi/2$ as $\omega \to \infty$. Various designs exists for filters that maintain the same asymptotic behavior but have different fall-off and phase behavior, particularly near $\omega RC \approx 1$.

RESONANCE AND THE TWO-POLE LC FILTER

Our next example uses a single LC pair, rather than 2 RC pairs, i.e.,



Kirchoff's Law gives

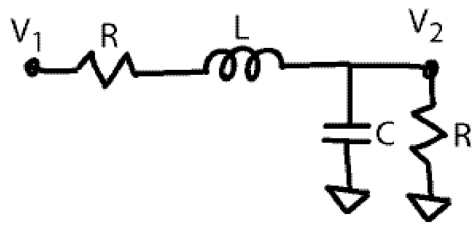
$$\frac{V_2 - V_1}{i\omega L} + V_2(i\omega C) = 0$$

or

$$V_2 = \frac{1}{1-\omega^2 LC} V_1$$

This shows the previous frequency-dependent fall-off, i.e., $V_2 \xrightarrow{\omega\to\infty} \frac{e^{-i\pi}}{\omega^2 LC} V_1$, but also shows resonance at ω^2LC = 1, which is not desirable. This can be tempered by the addition of a small resistance in series with the inductor, and/or a large resistance in parallel with the capacitor.

We will consider a special case of equal resistors and see under what conditions we can dampen the resonance. The circuit diagram is:



By working in terms of equivalent impedance, the analysis is straight forward.

The inductor/resistor pair has impedance $i\omega L + R = R(1 + i\omega L/R)$.

The capacitor/resistor pair as impedance $(R/i\omega C)/[R + (1/i\omega C)] = R/(1 + i\omega RC)$

Kirchoff's Law gives

$$\frac{V_2 - V_1}{R(1+i\omega L/R)} + V_2\left(\frac{1+i\omega RC}{R}\right) = 0$$

or

$$V_2 = \frac{1}{(1+i\omega RC)(1+i\omega L/R)+1} V_1 = \frac{1}{2-\omega^2 LC + i\omega(RC + L/R)} V_1 = \frac{2-\omega^2 LC - i\omega(RC+L/R)}{(2-\omega^2 LC)^2 + \omega^2 (RC + L/R)^2} V_1$$

or

$$V_2 = \frac{e^{-i\tan^{-1}\left[\frac{\omega(RC+L/R)}{2-\omega^2 LC}\right]}}{\sqrt{\left(2-\omega^2 LC\right)^2 + \omega^2\left(RC+L/R\right)^2}} V_1 \quad = \quad \frac{e^{-i\tan^{-1}\left[\frac{\omega(RC+L/R)}{2-\omega^2 LC}\right]}}{2\sqrt{1+\left(\omega^2 LC\right)^2 - \omega^2\left[LC - \left(\frac{RC+L/R}{2}\right)^2\right]}} V_1$$

from which we immediately see that the amplitude is finite at resonance when $\omega^2 LC = 1$.

A considerable simplification occurs for the choice $LC = \left(\frac{RC+L/R}{2}\right)^2$, which factors to $RC = L/R$ or $(RC)^2 = LC$, and drives the third term in the denominator to zero, i.e.,

$$V_2 = \frac{e^{-i\tan^{-1}\left[\frac{2\omega RC}{2-(\omega RC)^2}\right]}}{2\sqrt{1+(\omega RC)^4}} V_1 .$$

Note the complete loss of resonance; the factor of 2 in the denominator reflects voltage division by the resistors.

The above form must be contrast with that for the two-pole RC filter with buffering, for which

$$V_2 = \frac{e^{-i\tan^{-1}\left(\frac{2\omega RC}{1-(\omega RC)^2}\right)}}{1+(\omega RC)^2} V_1$$

They both have the same asymptotic limit, but the fall-off near $\omega = (RC)^{-1}$ is sharper for the two-pole RC filter than with the LCR filter. On the other hand, the LCR circuit does not require an Op-Amp.