Image Location and Magnification in Holography

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A method is presented for determining the location and magnification of images formed by holographic reconstruction. The method is based upon the similarity between a hologram and a zone plate. An undergraduate student familiar with introductory optics should be able to understand the method presented.

The concept that a hologram is a generalized zone plate was first put forward by Rogers[1] and has since been mentioned by many others. In spite of these publications the usefulness of the zone plate approach to holography is not appreciated widely. The purpose of this paper is to call attention once more to this concept and demonstrate its value in calculating the location and magnification of images produced by holography. The author believes that the approach presented here may be understood by the undergraduate student of optics more easily than he grasps the methods usually offered to him.

THE HOLOGRAM AS A ZONE PLATE

We begin by considering the in line or Gabor[2] type hologram produced by a single scattering point P as shown in Fig. 1. A photographic plate H is illuminated directly by a point source S and also by light from S which is scattered at the point P. Two coherent spherical wavefronts of radii u and v, respectively, fall upon the photographic plate H, which records the interference pattern produced. Upon development, the plate H becomes the hologram of the point object P. We consider the light reaching H directly from the source S as the reference wave.

Symmetry requires that the interference pattern have rotational symmetry about the axis (i.e., a line through S and P). It consists of concentric rings. A simple calculation shows that the radius r_m of the mth bright ring is related to the other quantities involved by the equation

$$\frac{1}{u}-\frac{1}{v}=\frac{2m\lambda}{r_m{}^2}\equiv\frac{1}{f}. \qquad (1)$$

In this equation the order number m is counted outward from the center, beginning with $m=0$. The wavelength of the light is represented by λ and the quantify f is defined by the equation above. We notice that f, as just defined, is the same equation as that giving the primary focal length of a zone plate, which is

$$|f|=r_m{}^2/2m\lambda \quad \text{or} \quad r_m=(2\lambda|f|)^{1/2}m^{1/2}. \qquad (2)$$

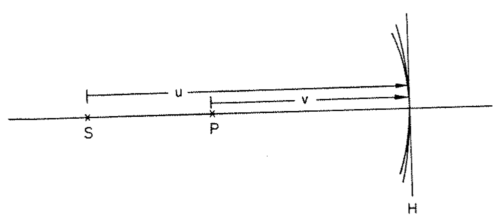

FIG. 1. Geometric arrangement to produce the hologram of a single point. S is the reference source; P is the scattering point. Spherical waves of radii u and v fall upon the hologram plane H. The hologram produced in this way is a zone plate centered on the line through S and P.

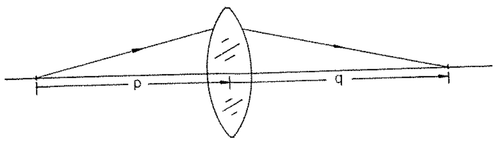

FIG. 2. The sign convention used with the thin lens equation. In this figure p, q, and f are all positive.

Here r_m is the radius of the mth transparent ring of a transparent centered zone plate or the mth opaque ring of an opaque centered zone plate. The absolute magnitude sign is necessary since a zone plate serves simultaneously as a positive (convergent) element and a negative (divergent) element. The developed hologram is a zone plate and its focal length as a zone plate can be calculated from Eq. (1). The zone plate has both positive and negative focal lengths given by $\pm|f|$. This hologram differs from the conventional zone plate in that the boundaries between light and dark regions are gradual rather than abrupt. It may be called a "sine wave" zone plate.

LOCATION OF IMAGES

The similarity of Eq. (1) to the thin lens equation is obvious; so the thin lens equation may be used to calculate the focal length of the hologram. The problem of sign convention immediately becomes important. We elect to use the sign convention defined by Fig. 2 and the comment that in this figure all quantities, i.e., p, q, and f, are positive. In this sign convention the thin lens equation, which also applies to zone plates, is

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}. \tag{3}$$

Identify the source of the reference wave with the object and the scattering point with the image. In this case $p=u$ and $q=-v$, i.e., $q<0$, for the geometry of Fig. 1. The thin lens Eq. (3) may be used in this way to calculate the focal length of the hologram or zone plate. For the geometry of Fig. 1, f will be negative. Zone plates have at least two focal lengths $|f|$ and $-|f|$.

We have outlined a very simple procedure for calculating the focal length of a hologram zone plate. This hologram, or a copy of it, will be used to reconstruct two images of the scattering point P. To increase the generality of the discussion we allow for the use of a new wavelength λ' in the reconstruction process. (We shall use primes on all quantities describing the reconstruction process.) We also assume that the hologram may have been magnified in the copying process before it is used for reconstruction. Let

$$\mu=\lambda'/\lambda$$

and

$$\eta\equiv r_m'/r_m.$$

r_m' is the magnified radius of the mth ring of the zone plate. This magnification and wavelength change gives a new focal length f' for the hologram:

$$f'=f(\eta^2/\mu). \tag{4}$$

The source of the reconstruction wave is at S' (Fig. 3), and the zone plate (by diffraction) produces two images, one at I_v' and the other at I_R'. The locations of these two images can be calculated using the thin lens equation

$$\frac{1}{p'}+\frac{1}{q'}=\frac{1}{f'}, \tag{5}$$

if we identify the source distance with p' and use $\pm|f'|$ for the focal length. Use of the value f' will give q_v' and locate I_v', the virtual image.

Changing the sign to $-f'$ will give q_R' and locate I_R'. The thin lens sign convention must be used with q_R' and q_v'. For the geometry shown in Fig. 1, f is negative and f' is also negative. The subscripts R and v stand for "real" and "virtual," since it often happens that the image at I_v' is

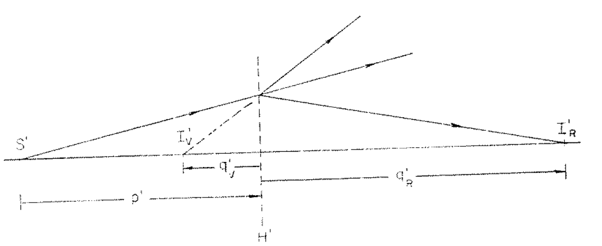

FIG. 3. The location of images formed in the reconstruction process. S' is the source of the reconstruction wave. I_v' is the "virtual" image and I_R' is the "real" image. H' is the hologram after development (and possible magnification).

virtual and the one at I_R' is real. It is not difficult to alter the experimental parameters p, q, and p' so that q_R' will be negative. In this case I_R' is no longer real but is still called I_R'.

It is also possible for I_v' to be real. For example, in Fig. 4 the reconstructing wavefront is converging toward S' which is to the right of the hologram. For this case the sign convention requires that p' be negative. If this is the same hologram as before (for which f and f' were negative) and if $|p'| < |f'|$, then I_v' will be real. Careful adherence to the sign convention throughout will take care of these problems, but I_v' is not always virtual nor is I_R' always real. Instead of "virtual" and "real," the terms "primary" and "conjugate" are often used.

We see, therefore, that the images formed in holography may be located by two applications of the thin lens equation: (1) First we ask what focal-length lens, if placed at the position of the photographic plate (undeveloped hologram), would image the source of the reference wave at the object. (2) If there is magnification of the hologram and/or a wavelength change before reconstruction, we use Eq. (4) to find a corrected value of the focal length. (3) We then locate the image of the reconstructing reference source as it would be formed by a lens of this focal length; this is the location of I_v'. The location of I_R' is determined by changing the sign of f' in step (3), which is the second application of the thin lens equation. The whole process is carried out using equations usually taught in introductory optics. If the same wavelength is used in making the hologram and reconstruction and if the original hologram is used (rather than a magnified copy), step (2) may be omitted.

If one delights in the introduction of new

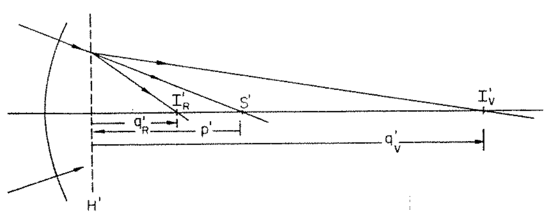

FIG. 4. Reconstruction with a convergent wavefront converging toward S'. For this condition $p'<0$. If $|p'| < |f'|$, I_v' will be positive and real as shown here.

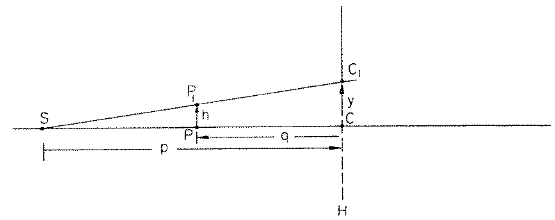

FIG. 5. The hologram of two points P and P_1 will consist of two overlapping zone plates centered at C and C_1, respectively. S is the source of the reference wave. The separation of P and P_1 is h; the separation of C and C_1 is y.

equations, Eqs. (3)–(5) may be put together so as to eliminate f and f'. The result is

$$q_v' = \eta^2 pp'q/(\mu pp' + \mu p'q - \eta^2 pq), \tag{6a}$$

$$q_R' = -\eta^2 pp'q/(\mu pp' + \mu p'q + \eta^2 pq). \tag{6b}$$

These are the equations usually presented,[3] but the notation and sign convention may vary.

MAGNIFICATION

If the object consists of two scattering points P and P_1, the hologram H will consist of two overlapping zone plates[4] one centered at C and the other at C_1, as indicated in Fig. 5. If h represents the separation of the object points and y represents the separation of the two centers C_1 and C, then by simple geometry

$$y = h[p/(p-|q|)] = h[p/(p+q)]. \tag{7}$$

After the hologram is developed and possibly magnified, the images are reconstructed as shown in Fig. 6. The center to center separation of the two zone plates is now $y' = \eta y$, where η is the linear magnification which the hologram may have experienced.

The reconstructed images of P are I_v' and I_R' and lie along the axis, which passes through C'. Their locations may be determined by the method of the previous section. The images of P_1 are I_{v1}' and I_{R1}'; these two points must lie along the line through S' and C_1' and are off axis by distances which we call h_v' and h_R'. The over-all magnification of the holographic process for these two images is defined as

$$M_v \equiv h_v'/h \quad \text{and} \quad M_R \equiv h_R'/h. \tag{8}$$

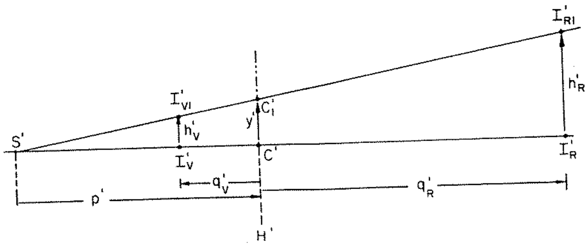

Fig. 6. In reconstruction the hologram of Fig. 5 gives images of P at I_v' and I_R'. Images of P_1 are formed at I_{v1}' and I_{R1}'.

From the similar triangles involved we conclude that

$$h_v' = y'(p' - |q_v'|)/p'$$
$$= y'(p' + q_v')/p'$$
$$= \eta y(p' + q_v')/p'$$
$$= \eta h[p/(p+q)][(p'+q_v')/p'].$$

The over-all magnification is

$$M_v \equiv h_v'/h = \eta[p/(p+q)](p'+q_v')/p'. \quad (9)$$

In like manner the real image has magnification,

$$M_R \equiv h_R'/h = \eta[p/(p+q)](p'+q_R')/p'. \quad (10)$$

The quantities q_v' and q_R' have been determined by the methods of the previous section. The other quantities on the right-hand side of these expressions for the image magnification are independent experimental variables.

In the special case in which the reference wave is plane ($p=\infty$) and the reconstructing wave is also plane ($p'=\infty$), these equations show that $M_v = M_R = \eta$. If the initial hologram is used rather than a magnified copy of it, $\eta = 1$; and both images have unit magnification. This conclusion is independent of μ and shows that for this case (i.e., $p=p'=\infty$) no magnification can be obtained by changing the wavelength between making the hologram and using it to reconstruct the images.

If one wishes he may substitute into Eqs. (9) and (10) the values of q_v' and q_R' as given in Eqs. (6a) and (6b). The results may be written in a variety of forms, among them

$$M_v = \frac{\eta}{1+(q/p)-(\eta^2 q/\mu p')} \quad (11a)$$

and

$$M_R = \frac{\eta}{1+(q/p)+(\eta^2 q/\mu p')}. \quad (11b)$$

These equations are found in the literature[3] in this form; again one must be alert to changes in the sign convention. In case $\eta=\mu=1$, a fairly common situation, the equations are somewhat simpler.

In the author's opinion Eqs. (6) and (11) are unnecessary complications. They are presented here to show that the results given by Eqs. (3)–(5) and by Eqs. (9) and (10) are equivalent to expressions currently found in the literature.

The author believes that the approach given here will take some of the mystery from holography and show the student that it is simply a two-step imaging process, a process which employs principles already familiar to the student.

The arguments have been presented for Gabor or "in line" holograms. Since an "off-axis" or Leith–Upatnieks[5] hologram may be considered as an off axis portion of an "in line" hologram, the arguments are still valid. However, the thin lens equations is taken from "paraxial" or "first-order" optics in which approximations equivalent to $\sin\theta = \theta = \tan\theta$ are used. For off-axis angles large enough to invalidate these approximations some error will result but the equations are still useful. Anyone wishing more accurate expressions for use when the off-axis angles are large, should consult the work of Champagne[6] or Lukosz.[7]

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[3] R. J. Collier, C. B. Burkhardt, and L. H. Lin, *Optical Holography* (Academic, New York, 1971), Chap. 3, pp. 71, 72.
[4] W. J. Siemens-Wapniarski and M. P. Givens, Appl. Opt. **7**, 535 (1968) or M. P. Givens, Amer. J. Phys. **35**, 1056 (1967).
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