Notes on the Critically Damped Harmonic Oscillator

Physics 2BL - David Kleinfeld

We often have to build an electrical or mechanical device. An understanding of physics may help in the design and tuning of such a device. Here, we consider a critically damped spring oscillator as a model design for the shock absorber of a car.

We consider a mass, denoted $m$, that is connected to a spring with spring constant $k$, so that the restoring force is $F = -kx$, and which moves in a lossy manner so that the frictional force is $F = -bv = -b\dot{x}$. Prof. Newton tells us that

$$\sum F = m\ddot{x} = -kx - b\dot{x}$$  \hspace{1cm} (1)

Thus

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0$$  \hspace{1cm} (2)

The two reduced constants are the natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$  \hspace{1cm} (3)

and the decay constant

$$\alpha = \frac{b}{m}$$  \hspace{1cm} (4)

so that we need to consider

$$\ddot{x} + \omega_0^2x + \alpha\dot{x} = 0$$  \hspace{1cm} (5)

The above equation describes simple harmonic motion with loss. It is discussed in lots of text books, but I want to consider a formulation of the solution that is most natural for critical damping.

We know that when the damping constant is zero, i.e., $\alpha = 0$, the solution of $\ddot{x} + \omega_0^2x = 0$ is given by:

$$x(t) = Ae^{i\omega_0t} + Be^{-i\omega_0t}$$  \hspace{1cm} (6)

where $A$ and $B$ are constants that are found from the initial conditions, i.e., $x(0)$ and $\dot{x}(0)$. In a nut shell, the system oscillates forever.
We know that when the natural frequency is zero, i.e., $\omega_0 = 0$, the solution of $\ddot{x} + \alpha \dot{x} = 0$ is given by:

$$\dot{x}(t) = Ae^{-\alpha t}$$ \hspace{1cm} (7)

and

$$x(t) = A \frac{1 - e^{-\alpha t}}{\alpha} + B$$ \hspace{1cm} (8)

where $A$ and $B$ are constants that are found from the initial conditions. In a nutshell, the system grinds to a halt.

A parenthetical remark, of relevance in the laboratory exercises, is that in the presence of a constant force field, like gravity, the main equation becomes $\ddot{x} + \alpha \dot{x} + g = 0$ and the solution simply picks up a constant to become

$$\dot{x}(t) = Ae^{-\alpha t} - \frac{g}{\alpha}$$ \hspace{1cm} (9)

In a nutshell, the system reaches a terminal velocity of

$$\dot{x}(t \leftarrow \infty) = \frac{mg}{b}$$ \hspace{1cm} (10)

on the time-scale of $t >> \alpha^{-1}$.

To return to the general case, we see that the presence of a decay term leads to an exponential loss in the amplitude of the system. It is natural to suppose that the damped oscillator has a solution of the form

$$x(t) = e^{-\beta t}u(t)$$ \hspace{1cm} (11)

where $\beta$ is a constant and we suspect that $u(t)$ may be the solution to an undamped harmonic oscillator. We can test this idea by computing derivatives and substituting them back into the original equation. We have

$$\dot{x}(t) = -\beta e^{-\beta t}u(t) + e^{-\beta t}\dot{u}(t)$$ \hspace{1cm} (12)

and

$$\ddot{x}(t) = -\beta e^{-\beta t}(\dot{u}(t) - \beta u(t)) + e^{-\beta t}[\ddot{u}(t) - \beta \dot{u}(t)]$$ \hspace{1cm} (13)

$$= e^{-\beta t}[\ddot{u}(t) - 2\beta \ddot{u}(t) + \beta^2 u(t)]$$
Thus

\[ e^{-\beta t} \left[ \ddot{u}(t) - 2\beta \dot{u}(t) + \beta^2 u(t) + \omega_0^2 u(t) + \alpha \dot{u}(t) - \alpha \beta u(t) \right] = 0. \]  \hspace{1cm} (14)

Since the prefactor \( e^{-\beta t} \) is never zero, the term in the brackets must be zero. This term simplifies considerably when the factors in front of the \( \dot{u}(t) \) terms sum to zero, which occurs for the choice

\[ \beta = \frac{\alpha}{2} \]  \hspace{1cm} (15)

Then we have

\[ \ddot{u}(t) + \left[ \omega_0^2 - \left( \frac{\alpha}{2} \right)^2 \right] u(t) = 0 \]  \hspace{1cm} (16)

which is the equation for simple harmonic motion with a frequency given by

\[ \omega = \sqrt{\omega_0^2 - \left( \frac{\alpha}{2} \right)^2} \]  \hspace{1cm} (17)

so that for \( \omega \neq 0 \)

\[ u(t) = Ae^{i\omega t} + Be^{-i\omega t} \]  \hspace{1cm} (18)

or

\[ x(t) = e^{-\frac{\alpha}{2} t} \left[ Ae^{i\sqrt{\omega_0^2 - \left( \frac{\alpha}{2} \right)^2} t} + Be^{-i\sqrt{\omega_0^2 - \left( \frac{\alpha}{2} \right)^2} t} \right] \]  \hspace{1cm} (19)

When \( \omega_0 > \frac{\alpha}{2} \), \( \omega \) is real and the solution has an oscillatory component. This is called the underdamped solution. When \( \omega_0 < \frac{\alpha}{2} \), \( \omega \) is imaginary and the solution is an exponential decay. This is called the overdamped solution.

The interesting case for us is when \( \omega_0 = \frac{\alpha}{2} \), so that

\[ \ddot{u}(t) = 0 \]  \hspace{1cm} (20)

The solution is

\[ u(t) = A + Bt \]  \hspace{1cm} (21)

so that

\[ x(t) = e^{-\frac{\alpha}{2} t} [A + Bt] \]  \hspace{1cm} (22)

This is denoted critical damping. In terms of the initial conditions

\[ x(t) = e^{-\frac{\alpha}{2} t} \left[ x(0) \left( 1 + \frac{\alpha}{2} t \right) + \dot{x}(0)t \right]. \]  \hspace{1cm} (23)
To simply matters in graphing $x(t)$, we take $\dot{x}(0) = 0$, so that

$$x(t) = x(0)e^{-\frac{\alpha}{2}t} \left(1 + \frac{\alpha}{2}t\right),$$  \hspace{1cm} (24)

and

$$\dot{x}(t) = -x(0)\frac{\alpha}{2} e^{-\frac{\alpha}{2}t} \left(\frac{\alpha}{2}t\right)$$  \hspace{1cm} (25)

and

$$\ddot{x}(t) = x(0) \left(\frac{\alpha}{2}\right)^2 e^{-\frac{\alpha}{2}t} \left(\frac{\alpha}{2}t - 1\right)$$  \hspace{1cm} (26)

Before we set out to graph the above kinematic variables, we note that

$$x(t \leftarrow 0) = x(0) + O(t^2),$$  \hspace{1cm} (27)

so that the slope of $x(t \leftarrow 0)$ is zero, i.e., $\dot{x}(0) = 0$. We also note that

$$\dot{x}(t \leftarrow 0) = -x(0)\frac{\alpha}{2} t + O(t^2)$$  \hspace{1cm} (28)

and

$$\ddot{x}(t \leftarrow 0) = -x(0) \left(\frac{\alpha}{2}\right)^2 + O(t),$$  \hspace{1cm} (29)

so that the system slows down with a constant deceleration from the very start. Lastly, we also see that the speed peaks at

$$t_{\text{max}} = \frac{2}{\alpha}.\hspace{1cm} (30)$$

In some sense, critical damping gives a "gentle" return to baseline.