PHYS 4L: Notes on Bode Plots

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1 Introduction

We are frequently interested in the **frequency response** of our circuits. Given an input sinusoidal waveform of some frequency, the frequency response tells us how much the output waveform will be amplified/attenuated ($A(\omega)$) as well as its relative phase shift ($\phi(\omega)$) from the input.

$$V_{in} = V_0 \sin(\omega t) \longrightarrow V_{out} = V_0 A(\omega) \sin(\omega t + \phi(\omega))$$

Thus, the frequency response characterizes the circuit's behavior in the frequency domain. If the system is linear and time-invariant, we can then use our frequency response to predict the circuit output of input waveforms of any shape. After all, we can transform any input waveform into the frequency domain with the Fourier Transform, which tells us about the frequency components which make up the input. We then multiply the frequency response with the input waveform's frequency domain representation. The magnitude of the frequency response scales the amplitudes of the input frequencies; meanwhile, the multiplication of the complex phase factors will sum the phase, shifting the original frequencies. Thus, we now have the properly scaled and phase shifted output frequencies, which we can inverse Fourier transform to the output waveform.

Of course, the entire analysis above can also be done in the time domain for simple circuits, but quickly becomes difficult to do for more complicated systems. Meanwhile, the frequency response is something which can be experimentally measured, as is done in Lab 2 for the simple RC low-pass and high-pass filter circuits.

2 Electrical Impedance

Motivated by the notion of a frequency response, it should be clear that we ought to consider the Fourier transform representation of our passive circuit elements.

The simplest case to treat is that of resistors, which are governed by Ohm's Law $V(t) = I(t)R$, where R is a real-valued constant. When we take the Fourier Transform, very little changes: $\tilde{V}(\omega) = \tilde{I}(\omega)R$.

Inductors and capcacitors are instead governed by differential equations in time. Here, the Fourier transform becomes very handy. Consider some function $f(t)$ represented as an inverse Fourier transform of its frequency representation:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{f}(\omega)d\omega \qquad \tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt$$

We can take the time derivatives of both sides of the equation on the left. As the integral is of ω, we can bring the derivative under the integral sign and differentiate. As only the complex exponential has time dependence, the equation can be written as follows:

$$\frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} i\omega\tilde{f}(\omega)d\omega$$

Note that this is written in the form of an inverse Fourier transform! Thus, whatever is in the integrand that is not the complex exponential must be the Fourier transform of the left-hand side. We then know that the Fourier transform of the time derivative of some function is $i\omega$ times said function's Fourier transform. By instead calling $df/dt = g(t)$ and $f(t) = \int g(t)dt$, we can also immediately find the Fourier transform of an integral. Summarized below:

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{f}(\omega) \qquad \frac{df(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} i\omega\tilde{f}(\omega) \qquad \int f(t)dt \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{i\omega}\tilde{f}(\omega)$$

Let us now return to the discussion concerning the inductor and capacitior. Recall that the former is governed by $V(t) = L\frac{dI(t)}{dt}$ while the latter is given by $V(t) = \frac{Q}{C} = \frac{1}{C}\int I(t)dt$. Below, we show the time domain and corresponding frequency domain representations:

$$V(t) = I(t)R \longleftrightarrow \tilde{V}(\omega) = I(\omega)R$$
$$V(t) = L\frac{dI(t)}{dt} \longleftrightarrow \tilde{V}(\omega) = \tilde{I}(\omega)i\omega L$$
$$V(t) = \frac{1}{C}\int I(t)dt \longleftrightarrow \tilde{V}(\omega) = \tilde{I}(\omega)\frac{1}{i\omega C}$$

In frequency space, all of the equations are of the form of Ohm's Law, but with complex valued "resistances". These are what we call impedances. We now generalize Ohm's Law for complex impedances as $V = IZ$, where Z is the impedance of whatever circuit element. Because these impedances are governed by Ohm's Law, we can solve circuits in the frequency domain as though every element is a resistor, and just use Kirchoff's laws as always. As an additional result, impedances add in series and in parallel like resistors.

$$Z_R = R \qquad Z_L = i\omega L \qquad Z_C = \frac{1}{i\omega C}$$

3 Passive First Order Filters

Most of the filters presented in Lab 2 are the simplest possible electronic circuits, consisting of only two passive circuit elements. The optional LRC circuit is an example of a higher order passive filter which achieves a much steeper attenuation after passing its f_{3dB} point. Later in the course, we will study circuits with active components which can also be similarly analyzed but can additionally achieve amplification rather than just attenuation.

In these notes, I will consider the example LR circuit shown below.

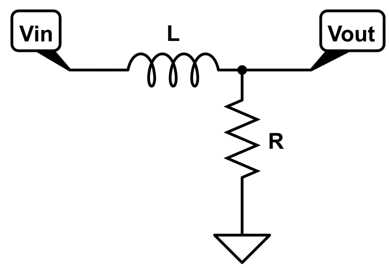

Figure 1: LR first order low-pass filter. $L = 94\text{mH}$ and $R = 9.9\text{k}\Omega$.

Prior to any advanced circuit analysis, we can immediately tell this is a low-pass filter. In the extreme case of $\omega \to 0$, the inductor's impedance $i\omega L \to 0$, which is a short-circuit (wire). Thus, V_{out} is on the same node as V_{in}, so they must be equal at low frequency. On the other hand, for $\omega \to \infty$, the inductor impedance goes as $i\omega L \to \infty$, which is an open-circuit. In that case, the circuit is broken so no current flows. As a result, there is no voltage drop across the resistor, so V_{out} gets pulled to ground at high frequency. Summarizing, this circuit passes V_{in} to V_{out} for low-frequencies and attenuates to 0 at high-frequencies - it is a **low-pass** filter.

3.1 Theory: Transfer Function

We will now find analytical expressions for the magnitude and phase of the frequency response. To do this, let us first find the relation between $\tilde{V}_{out}$ and $\tilde{V}_{in}$. By recognizing that our circuit is simply a voltage divider of complex impedances, we can immediately write the result:

$$\tilde{V}_{out}(\omega) = \frac{Z_R}{Z_R + Z_L}\tilde{V}_{in}(\omega)$$

Aside: Review of Voltage Dividers

Being able to recognize voltage dividers quickly is a crucial skill to develop in this course! Thus, we will briefly review it. We can generalize the circuit shown in Figure 1 to any two passive elements connected in series with impedances Z_1 and Z_2, configured such that Z_1 takes the place of the inductor and Z_2 replaces the resistor. The total impedance is therefore $Z_T = Z_1 + Z_2$. Using Ohm's Law, the current flowing through the circuit must be $I = \tilde{V}_{in}/Z_T$. As the components are in series, the current through either of the components is still $I = \tilde{V}_{in}/Z_T$. By noting that $\tilde{V}_{out}$ is simply the voltage drop across Z_2, we use Ohm's Law to find $\tilde{V}_{out} = IZ_2 = \frac{Z_2}{Z_T}\tilde{V}_{in}$.

Rearranging our voltage divider equation yields the frequency response, which I will henceforth denote as $H(\omega)$:

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{R}{R + i\omega L}$$

It is additionally convenient to define quantity $\tau = L/R$, which has units of time, yielding:

$$H(\omega) = \frac{1}{1 + i\omega\tau} = \frac{1 - i\omega\tau}{1 + (\omega\tau)^2}$$

The magnitude and phase of the frequency response are found the usual way for any complex-valued function, resulting in:

$$A(\omega) = |H(\omega)| = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \qquad \phi(\omega) = \angle(H(\omega)) = -\arctan(\omega\tau)$$

Notice that $\omega\tau$ is unitless. Taking the limits $\omega\tau \ll 1$ and $\omega\tau \gg 1$ correspond to to taking the low and high frequency limits - think about why this is. As a hint, recall that the f_{3dB} value is related to τ. Taking these limits will also allow understanding of linear approximations of the resulting Bode plots for the low and high frequency limit - be sure to do this and explain the results on your lab report!

Finally, putting everything together, we can write the output for any given input:

$$V_{in}(t) = V_0 \sin(\omega t) \longrightarrow V_{out}(t) = \frac{V_0}{\sqrt{1 + (\omega\tau)^2}} \sin(\omega t - \arctan(\omega\tau))$$

3.2 Experiment: Measured Quantities

From the above discussion, the experimental procedure should be clear. The ratio of the output amplitude $V_0/\sqrt{1 + (\omega\tau)^2}$ to the input amplitude V_0 yields the magnitude of the frequency response. Similarly, the phase shift of the frequency response is simply how much the output signal is shifted from that of the input signal.

We now only need to measure these quantities over 20 logarithmically even steps in frequency across 2 decades, centered about f_{3dB}. If $f = 10^x$ is logarithmically evenly space, exponent x is linearly even. The starting and ending values of x are then simply $a = \log_{10}(0.1f_{\text{-}3dB})$ and $b = \log_{10}(10f_{\text{-}3dB})$. For a total of N = 20 points, we have N-1 = 19 evenly spaced steps, so the exponent step size is $\frac{b-a}{19}$, allowing us to define each frequency we ought to measure.

3.3 Bode Plots

To visualize the frequency response, we plot the information on a Bode plot. As changes typically happen over multiple orders of magnitude of frequency, the x-axis of Bode plots are logarithmically scaled. The Bode magnitude plot tells us how much the frequency response amplifies/attenuates the signal, $A(\omega) = |\tilde{V}_{out}/\tilde{V}_{in}|$. This quantity is generally plotted in units of decibels (dB), which is $20\log_{10}|A(\omega)|$. On the other hand, the Bode phase plot tells us how much the frequency response phase shifts the signal. Below is an example Bode plot for the circuit analysis done above, using the measured values of $R = 9.9\text{k}\Omega$ and $L =$94mH.

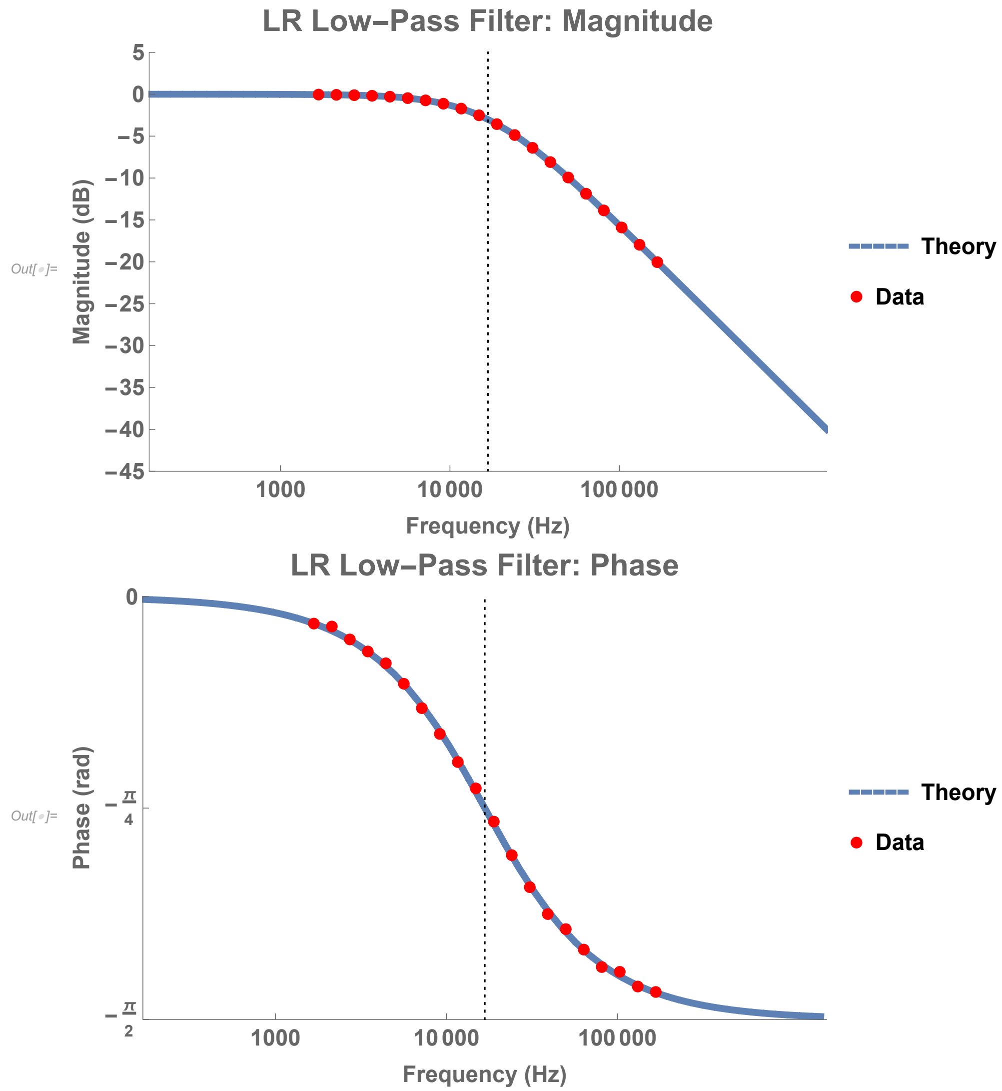

Figure 2: LR first order low-pass filter Bode plot of magnitude (above) and phase (below). Theory is plotted as a blue curve, while experimental data points are plotted as red markers. The vertical dotted line is the theoretically predicted f_{3dB} value.