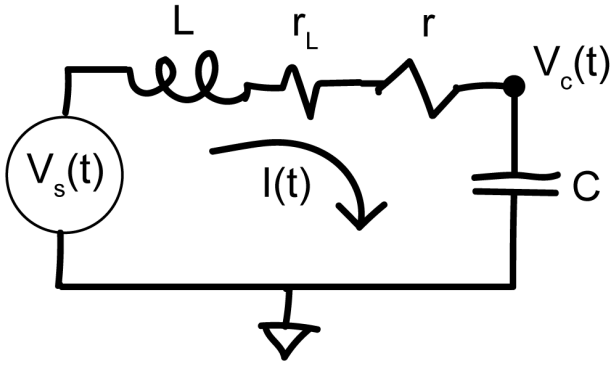
4 Week 4 - The RLC damped oscillator

We consider the three regions of the damped oscillator for a series RLC circuit (Figure 1). The series configuration was chosen for simplicity of analysis, given that the inductor has a non-negligible internal resistance.

Figure 1: RLC circuit with elements in series



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4.1 Formulation

The application of Kirchoff's voltage law yields

$$0 = -V_s(t) + L\frac{dI(t)}{dt} + r_L I(t) + rI(t) + \frac{1}{C}\int_{-\infty}^{t} dt'\ I(t'). \tag{4.4}$$

We let $R = r + r_L$, the total resistance. Since we will measure voltage in the laboratory, we can put this in terms of the voltage drop across the capacitor, i.e., $V_C(t)$

$$I(t') = C\frac{dV_C(t)}{dt}. \tag{4.5}$$

Then

$$V_s(t) = LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) \tag{4.6}$$

There are two time scales. One is our old friend, the time constant of an RC circuit, i.e.,

$$\tau_C \equiv RC \tag{4.7}$$

The second will become identified with the natural frequency of the circuit when $R = 0$, i.e., lossless, and is

$$\omega_o \equiv \frac{1}{\sqrt{LC}} \tag{4.8}$$

Then

$$V_s(t) = \frac{1}{\omega_o^2}\frac{d^2V_C(t)}{dt^2} + \tau_C\frac{dV_c(t)}{dt} + V_C(t) \tag{4.9}$$

For completeness, recalling that

$$\tau_L \equiv \frac{L}{R}, \tag{4.10}$$

so we have

$$\tau_C\tau_L = \frac{1}{\omega_o^2} \tag{4.11}$$

and we get back to the typical form of the equation for a damped oscillator:

$$\omega_o^2 V_s(t) = \frac{d^2V_C(t)}{dt^2} + \frac{1}{\tau_L}\frac{dV_c(t)}{dt} + \omega_o^2 V_C(t) \tag{4.12}$$

where damping is caused by the resistive loss of energy.

4.1.1 Homogeneous solution

The damped oscillator equation describes simple harmonic motion with loss. It is discussed in lots of text books. We consider a formulation of the solution that is natural for understanding the transition from an over-damped oscillator to an under-damped oscillator that highlights the special case of critical damping,

Let's look at the homogeneous equation. We know that when the damping is zero, i.e., $R = 0$ so $\tau_C = 0$ or $\tau_L = \infty$, the solution of

$$0 = \frac{d^2V_C(t)}{dt^2} + \omega_o^2 V_C(t) \tag{4.13}$$

is

$$V_C(t) = Ae^{+i\omega_o t} + Be^{-i\omega_o t} \tag{4.14}$$

where A and B are constants that are found from the initial conditions, i.e., $V_C(0)$ and $I(0)$. In a nut shell, the system oscillates forever.

We know that when the damping is strong, i.e., $R \to \infty$, so $\tau_C \gg \tau_L$, the solution of

$$0 = \frac{dV_C(t)}{dt} + \frac{1}{\tau_C}V_C(t) \tag{4.15}$$

is given by:

$$V_C(t) = Ae^{-t/\tau_C} + B \tag{4.16}$$

where A and B are constants that are found from the initial conditions. In a nut shell, the system grinds to a halt.

The damped solution motivates the choice of an an exponential decay as an integrating factor, for which we assume a solution of the form

$$V_C(t) = e^{-\beta t}u(t) \tag{4.17}$$

where β is a positive constant that we need to find self-consistently and $u(t)$ will have two solution because of the second-order nature of our linear equation. To proceed, we have

$$\begin{aligned} \frac{dV_C(t)}{dt} &= -\beta e^{-\beta t}u(t) + e^{-\beta t}\frac{du(t)}{dt} \\ &= e^{-\beta t}\left[\frac{du(t)}{dt} - \beta u(t)\right] \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \frac{d^2V_C(t)}{dt^2} &= -\beta e^{-\beta t}\left[\frac{du(t)}{dt} - \beta u(t)\right] + e^{-\beta t}\left[\frac{d^2u(t)}{dt^2} - \beta\frac{du(t)}{dt}\right] \\ &= e^{-\beta t}\left[\frac{d^2u(t)}{dt^2} - 2\beta\frac{du(t)}{dt} + \beta^2 u(t)\right] \end{aligned} \tag{4.19}$$

Thus

$$0 = e^{-\beta t}\left[\frac{d^2u(t)}{dt^2} + \left(\frac{1}{\tau_L} - 2\beta\right)\frac{du(t)}{dt} + \left(\beta^2 - \frac{\beta}{\tau_L} + \omega_0^2\right)u(t)\right] \tag{4.20}$$

Since the prefactor $e^{-\beta t}$ is never zero, the term in the brackets must be zero. This term simplifies considerably when the factors in front of the $du(t)/dt$ term sum to zero, which occurs for the choice

$$\beta = \frac{1}{2\tau_L}. \tag{4.21}$$

This leads to

$$\ddot{u}(t) + \left[\omega_o^2 - \left(\frac{1}{2\tau_L}\right)^2\right]u(t) = 0, \tag{4.22}$$

which is the equation for simple harmonic motion with a frequency given by

$$\omega = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau_L}\right)^2}. \tag{4.23}$$

We see that the effect of damping is to lower the frequency of oscillation from ω_o toward zero. For $\omega \neq 0$

$$u(t) = Ae^{+i\omega t} + A^*e^{-i\omega t} \tag{4.24}$$

where A is complex, or

$$V_C(t) = e^{-t/2\tau_L}\left[Ae^{+i\omega t} + A^*e^{-i\omega t}\right] \tag{4.25}$$

4.1.2 Underdamped solution

When $\omega_o > \frac{1}{2\tau_L}$, ω is real and the solution has a damped oscillatory component. Further, when the damping is weak, the frequency is given by

$$\omega \simeq \omega_o\left[1 - \frac{1}{8}\left(\frac{1}{\omega_o\tau_L}\right)^2 + \cdots\right]. \tag{4.26}$$

4.1.3 Overdamped solution

When $\omega_o < \frac{1}{2\tau_L}$, ω is imaginary and the solution is solely damped. Thus

$$\omega = i\frac{1}{2\tau_L}\sqrt{1-(2\omega_o\tau_L)^2}. \tag{4.27}$$

This leads to two damping times, i.e.,

$$\tau_{1,2} = \frac{1}{2\tau_L}\left[1 \mp \sqrt{1-(2\omega_o\tau_L)^2}\right] \tag{4.28}$$

When the damping is strong, this reduces to

$$\tau_{1,2} = \frac{1}{\tau_L} \text{ and} \frac{1}{\tau_C}. \tag{4.29}$$

4.1.4 Critical damped solution

An interesting case is when $\omega_0 = \frac{1}{2\tau_L}$, or in terms of components $R = 2\sqrt{\frac{L}{C}}$, so that

$$\frac{d^2u(t)}{dt^2} = 0 \tag{4.30}$$

for which

$$u(t) = A + Bt \tag{4.31}$$

so that

$$V_C(t) = e^{-\frac{t}{2\tau_L}}\left[A + Bt\right] \tag{4.32}$$

and

$$\begin{aligned} I_C(t) &= C\frac{dV_C(t)}{dt} \\ &= e^{-\frac{t}{2\tau_L}}\left[-\frac{A}{2\tau_L} + B\left(1 - \frac{t}{2\tau_L}\right)\right]. \end{aligned} \tag{4.33}$$

In terms of the initial conditions

$$V_C(t) = e^{-\frac{t}{2\tau_L}}\left[V_C(0)\left(1 + \frac{1}{2\tau_L}t\right) + \frac{I_C(0)}{C}t\right]. \tag{4.34}$$

For the case of $I(0) = 0$, which is typical, the voltage changes "gently" after a step. That is

$$V_C(t \ll \tau_L) = V_C(0)\left[1 - \left(\frac{t}{2\tau_L}\right)^2 + \cdots\right] \tag{4.35}$$

so the decay is parabolic, rather than sharp.

4.1.5 Phase analysis of the undamped driven oscillator

Let's look back at the at the case when the damping is zero, i.e., $R = 0$ so $\tau_C = 0$ or $\tau_L = \infty$, but with an external sinusoidal drive $\ddot{V}_D \sin \omega_D t$. This will allow us to understand how the circuit behaves near resonance, i.e., as a function of the difference in drive versus resonant frequency.

$$\ddot{V}_D \sin \omega_D t = \frac{d^2 V_C(t)}{dt^2} + \omega_o^2 V_C(t). \tag{4.36}$$

The steady-state solution for $V_C(t)$ will be at the drive frequency ω_D and we safely postulate that it is of the form

$$\begin{aligned} V_C(t) &= A \sin(\omega_D t + \phi) \\ &= A \cos\phi \sin(\omega_D t) + A \sin\phi \cos(\omega_D t) \end{aligned} \tag{4.37}$$

where A and ϕ are constants. Then

$$\begin{aligned} \ddot{V}_D \sin \omega_D t &= -\omega_D^2 A \cos\phi \sin(\omega_D t) - \omega_D^2 A \sin\phi \cos(\omega_D t) + \omega_o^2 A \cos\phi \sin(\omega_D t) + A \sin\phi \cos(\omega_D t) \\ &= -A(\omega_o^2 - \omega_D^2) \cos\phi \sin(\omega_D t) - A(\omega_o^2 - \omega_D^2) \sin\phi \cos(\omega_D t). \end{aligned} \tag{4.38}$$

Thus

$$0 = (\omega_o^2 - \omega_D^2) \sin\phi, \tag{4.39}$$

which implies that, except for exactly at resonance where $\omega_o = \omega_D$, the phase takes on $\phi = 0$ or $\phi = \pi$. Further

$$\ddot{V}_D = A(\omega_o^2 - \omega_D^2) cos\phi \tag{4.40}$$

so that $\phi = 0$ for $\omega_D < \omega_o$ and $\phi = \pi$ for $\omega_D > \omega_o$. In other words, the phase of the output shifts by π radians as the drive frequency passes through the resonant frequency. The complete solution is

$$V_C(t) = \left\{ \begin{array}{ll} +1 & \text{if } \omega_D < \omega_o \\ -1 & \text{if } \omega_D > \omega_o \end{array} \right\} \frac{\ddot{V}_D}{|\omega_o^2 - \omega_D^2|} \sin(\omega_D t) \tag{4.41}$$

This result holds when loss is considered, except that the phase shift occurs smoothly, rather than as a step, as the frequency is shifted from above to below ω_o.