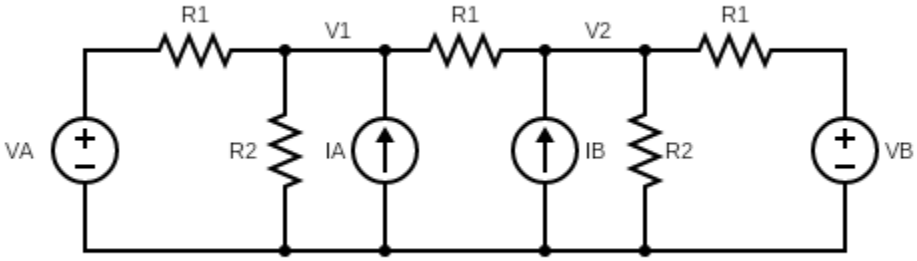
Homework 1 Solutions

Problem 1. Solve for V_1 and V_2.



We want to find expressions for the voltages at the labeled nodes. Let's go ahead and choose the bottom node to be grounded. We can apply Kirchoff's Current Law (KCL) at the two nodes:

KCL at V_1: $\frac{V_A - V_1}{R_1} + \frac{0 - V_1}{R_2} + \frac{V_2 - V_1}{R_1} + I_A = 0$

KCL at V_2: $\frac{V_B - V_2}{R_1} + \frac{0 - V_2}{R_2} + \frac{V_1 - V_2}{R_1} + I_B = 0$

If we go ahead and collect like terms, we get two equations:

1. $\left(\frac{2}{R_1} + \frac{1}{R_2}\right) V_1 + \left(-\frac{1}{R_1}\right) V_2 = I_A + \frac{V_A}{R_1}$
2. $\left(-\frac{1}{R_1}\right) V_1 + \left(\frac{2}{R_1} + \frac{1}{R_2}\right) V_2 = I_B + \frac{V_B}{R_1}$

We can now formulate the problem in matrix form:

$$\begin{bmatrix} \frac{2}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{2}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_A + \frac{V_A}{R_1} \\ I_B + \frac{V_B}{R_1} \end{bmatrix}$$

Let's do a sanity check. Notice that the 2x2 matrix has elements with units of inverse resistance, otherwise known as conductance. This conductance matrix is multiplied to a voltage vector to yield a resultant vector whose elements have units of current, as we would expect.

To solve, we apply the matrix inverse to both sides:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\left(\frac{2}{R_1} + \frac{1}{R_2}\right)^2 - \left(\frac{1}{R_1}\right)^2} \begin{bmatrix} \frac{2}{R_1} + \frac{1}{R_2} & \frac{1}{R_1} \\ \frac{1}{R_1} & \frac{2}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} I_A + \frac{V_A}{R_1} \\ I_B + \frac{V_B}{R_1} \end{bmatrix}$$

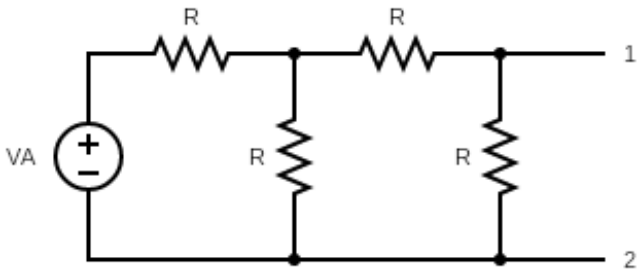
This therefore yields our solution:

$$V_1 = \frac{1}{\left(\frac{2}{R_1}+\frac{1}{R_2}\right)^2 - \left(\frac{1}{R_1}\right)^2}\left[\left(\frac{2}{R_1}+\frac{1}{R_2}\right)\left(I_A + \frac{V_A}{R_1}\right) + \frac{1}{R_1}\left(I_B + \frac{V_B}{R_1}\right)\right]$$

$$V_2 = \frac{1}{\left(\frac{2}{R_1}+\frac{1}{R_2}\right)^2 - \left(\frac{1}{R_1}\right)^2}\left[\frac{1}{R_1}\left(I_A + \frac{V_A}{R_1}\right) + \left(\frac{2}{R_1}+\frac{1}{R_2}\right)\left(I_B + \frac{V_B}{R_1}\right)\right]$$

Let us also notice that, if V_A=V_B and I_A=I_B, the circuit becomes symmetric across the middle such that the two voltages V_1 and V_2 should be identical. Indeed, the above boxed solutions reduce to the same expressions if the sources are identical.

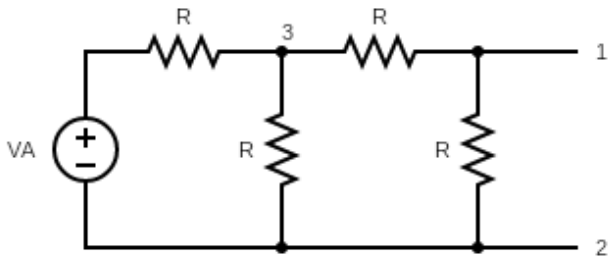
Problem 2. Find the Thevenin equivalent circuit between nodes 1 and 2:



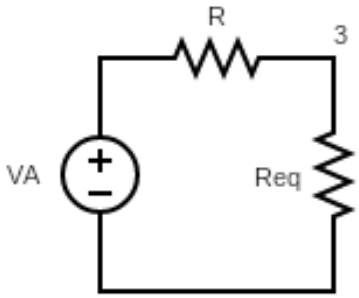
There are a few different approaches to solving this problem.

Method 1: Solve for open circuit voltage between nodes 1 and 2 for Thevenin's equivalent voltage. Then, "turn off" independent voltage or current sources and find the equivalent resistance between nodes 1 and 2 for Thevenin's equivalent resistance. Note: This method does not work if dependent power sources are present, but that is outside the scope of this course.

Let us go ahead and treat node 2 as ground. We can notice that the voltage at node A will simply be a voltage divider of the voltage at node 3, as labeled below. Thus, if we can find the voltage at node 3, we can directly find the open circuit voltage.



We can recognize that there are two resistors R in series on the path from nodes 3 to 2 to ground, which are in parallel to the resistor R from node 3 directly to ground. Thus, we can replace these resistors with an equivalent resistance as drawn below:



$$R_{3\to1\to GND} = R + R = 2R \qquad R_{3\to GND} = R$$

$$R_{eq} = \left(\frac{1}{R_{3\to1\to GND}} + \frac{1}{R_{3\to GND}}\right)^{-1} = \left(\frac{1}{2R} + \frac{1}{R}\right)^{-1} = \frac{2}{3}R$$

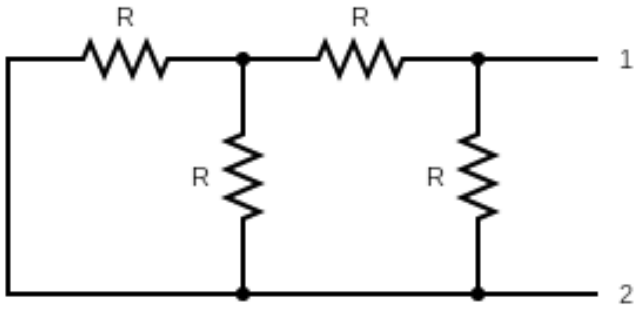
The voltage at node C is a voltage divider of the source voltage:

$$V_3 = \frac{R_{eq}}{R_{eq} + R} V_A = \frac{2}{5} V_A$$

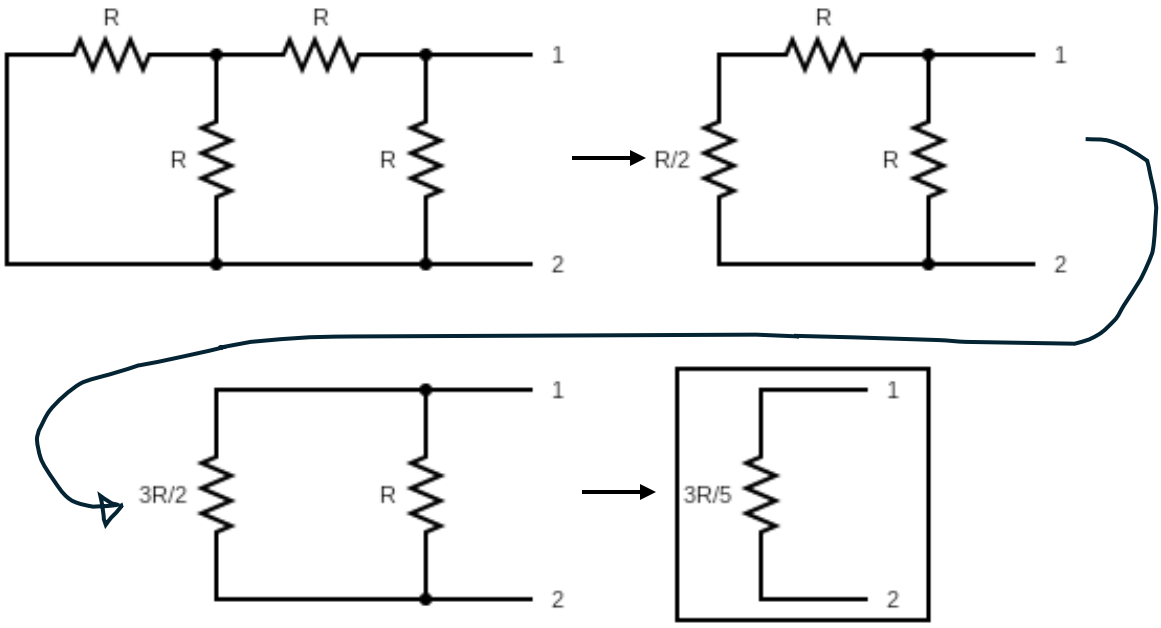
Thus, we find the Thevenin's equivalent voltage to be:

$$\boxed{V_{Th} = V_{OC} = V_2 = \frac{R}{R + R} V_C = \frac{V_A}{5}}$$

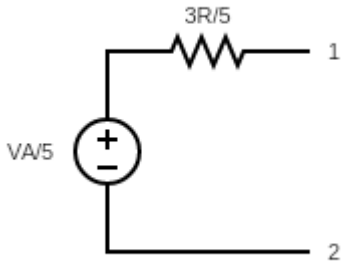
We can find the Thevenin's equivalent resistance by replacing the independent voltage source with its equivalent resistance, which is to say, we replace it with a short:



Our task then is to simplify by combining resistors either in parallel or in series until we are left with only one equivalent resistance between nodes 1 and 2. The process is shown below:

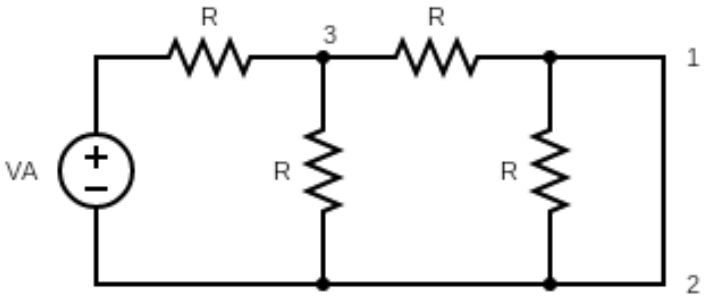


Having now found both the Thevenin's equivalent voltage and resistance, we can draw the equivalent circuit:

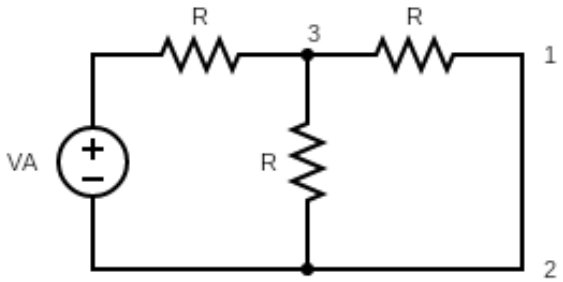


Method 2: Find the open circuit voltage for Thevenin's equivalent voltage just as in Method 1. Then find the short circuit current between nodes 1 and 2 for the Norton's equivalent current. Take the ratio via Ohm's Law to identify the Thevenin's equivalent resistance.

The short circuit current is found by placing a wire between nodes 1 and 2.



Notice that doing so means no current flows across the resistor R placed in parallel to the short, so we can remove it from the circuit.



Finding the short circuit current is therefore equivalent to finding the current through the resistor R between nodes 3 and 1. This is made trivial if we know the voltage at node 3, which we can again recognize to be that of a voltage divider of the source voltage. The equivalent resistance between node 3 and 2 is simply that of two resistors R in parallel, such that:

$$V_3 = \frac{R_{eq}}{R_{eq} + R} V_A = \frac{\frac{R}{2}}{\frac{R}{2} + R} V_A = \frac{1}{3} V_A$$

Then, we find the Norton's equivalent current as:

$$\boxed{I_N = I_{SC} = \frac{V_3}{R} = \frac{1}{3R} V_A}$$

The Thevenin's equivalent resistance is then simply the ratio between the Thevenin's equivalent voltage and the Norton's equivalent current, yielding:

$$\boxed{R_{Th} = \frac{V_{Th}}{I_N} = \frac{\frac{1}{5} V_A}{\frac{1}{3R} V_A} = \frac{3}{5} R}$$

Problem 3. Why should you check the voltage of a battery using a load?

Real batteries have non-zero internal resistances and can only supply so much power. The voltage across a given load will only approach the nominal value advertised by the battery if the load resistance is significantly larger than the internal resistance of the battery, such that the load does not draw huge currents from the battery. By inverting the voltage divider equation, we could even estimate the internal resistance of the battery.