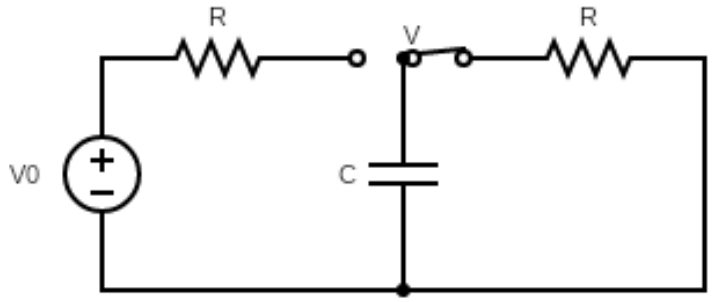
Homework 2 Solutions

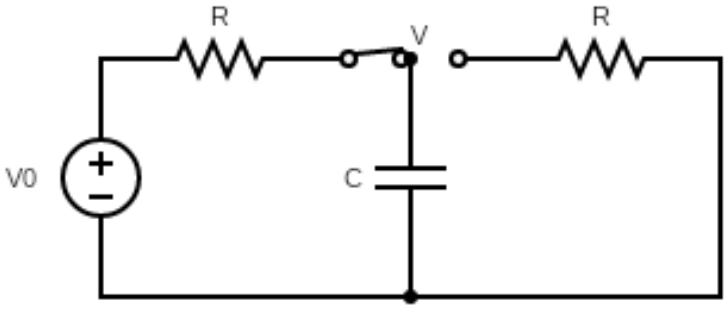
Problem 1. The switch is closed to the right for all time $t < 0$. What is the voltage, V, for $t < 0$?



The voltage V sits on capacitor C, which, for time $t < 0$, is connected to resistor R to ground and nothing else. Because the capacitor is in a purely passive network with no independent sources, there is nothing that can hold up a nonzero voltage indefinitely. Any charge which existed on the capacitor has a discharge path to ground and there is no place for charge to otherwise arrive on the capacitor. Thus,

$$\boxed{V(t < 0) = 0}$$

Problem 2. The switch is then closed to the left at time $t = 0$.



a) Immediately after closing the switch, what is V at $t = 0$?

Recall that the current through a capacitor is:

$$I_C = C\frac{dV}{dt}$$

If we assume that the current through the capacitor must be finite, then the time-derivative of the voltage across it must be finite as well. Thus, the voltage across the capacitor must be continuous in time. Otherwise, a discontinuity in the voltage would imply an infinite rate of change, leading to an infinite current coursing through the capacitor. In other words, the voltage across the capacitor cannot instantaneously change and will still be as it was before the switch flipped:

$$\boxed{V(0^+) = V(0^-) = 0}$$

b) What is V for $t \gg RC$?

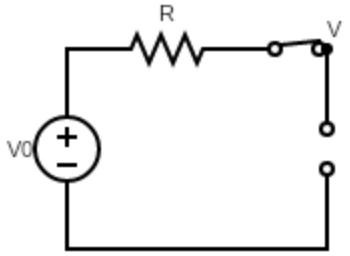
After the switch is flipped at $t = 0$, we wait for sufficiently long time such that the transient voltages die out with characteristic time scale $\tau = RC$. In this long-time limit, the circuit reaches steady state. In the absence of any time-dependent sources, the only voltages present in the steady state must be constant. Therefore

$$\left.\frac{dV}{dt}\right|_{t \gg RC} = 0$$

As the current through the capacitor scales with the time rate of change of the voltage across it, this implies that no current flows through the capacitor in DC steady state:

$$I_c = C\frac{dV}{dt} = 0$$

An element which passes no current regardless of the voltage drop across it is exactly an **open circuit**. Thus, the capacitor effectively becomes an open circuit in DC steady state, reducing our circuit to the one below.



Thus, no current flows in the circuit at all, and the voltage across the capacitor (now an open circuit) is simply that of the voltage source:

$$\boxed{V(t \gg RC) = V_0}$$

c) Write the expression for $V(t)$ for $t \geq 0$.

To solve, we note that the capacitor is in series with the resistor. Thus, the current through the capacitor is the current through the resistor:

$$I_c = C\frac{dV}{dt} = \frac{V_0 - V}{R} = I_R$$

Rearranging, we get a forced first-order differential equation:

$$V + RC\frac{dV}{dt} = V_0$$

In the specific case of a constant forcing term as we have in this situation, we can make a U-substitution of $U = V - V_0$, such that $\frac{dU}{dt} = \frac{dV}{dt} - \frac{dV_0}{dt} = \frac{dV}{dt}$, resulting in the homogenous first-order differential equation:

$$U + RC\frac{dU}{dt} = 0$$

We know that this is solved by a solution which goes as exponential decay with time constant RC:

$$U(t) = Ae^{-\frac{t}{RC}}$$

To find A, we plug in for time $t = 0$, which yields $U(0) = A$. We already know the initial condition:

$$U(0) = V(0) - V_0 = -V_0$$

As a result,

$$U(t) = -V_0 e^{-\frac{t}{RC}} = V(t) - V_0$$

We therefore find the expression for voltage to be:

$$\boxed{V(t \geq 0) = V_0\left(1 - e^{-\frac{t}{RC}}\right)}$$

Let us check that this solution makes sense. At time $t = 0$, we indeed have $V(0) = 0$, as previously found. On the other hand, for sufficiently long time, $V(t \gg RC) = V_0$, which we also expect. In between, the voltage exponentially decays from the initial voltage to its final voltage with time constant RC.

We could also have chosen to not substitute for the initial condition and steady state value. Doing so would have yielded the general solution for any DC forcing voltage:

$$V(t \geq 0) = \big(V(0) - V(t \gg RC)\big)e^{-\frac{t}{RC}} + V(t \gg RC)$$

We see that the general solution tells us the difference between the initial voltage and the steady state voltage decays away, leaving only the steady state voltage in the long term.

Problem 3. We then close the switch to its original position to the right at time $t = T \gg RC$.

a) What is the voltage V immediately after closing the switch at time $t = T$?

As with before, the voltage on the capacitor cannot change immediately. As the capacitor was previously given sufficient time to reach steady state, the voltage immediately after closing the switch stays the same as our solution in problem 2b.

$$\boxed{V(T^+) = V(T^-) = V_0}$$

b) What is the voltage V at time $t \gg T + RC$, long after the switch has been closed back?

Via the same arguments as those made in problem 1, the capacitor can only discharge, so it ends up as the same voltage as before the switch was ever closed to the left.

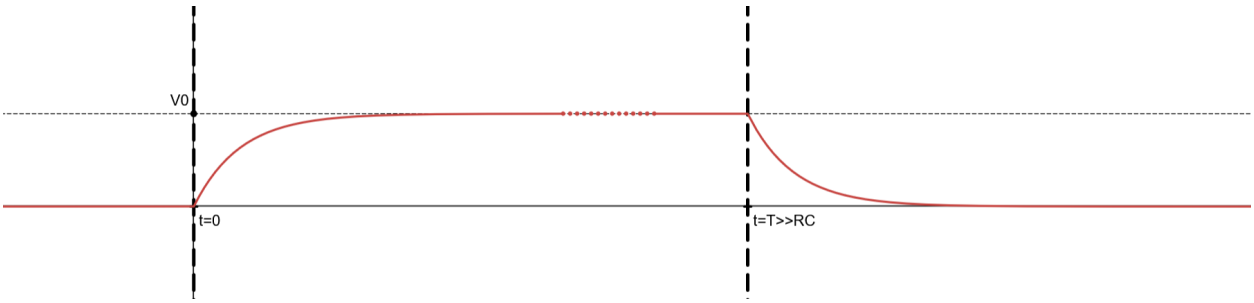
$$\boxed{V(t \gg T + RC) = 0}$$

c) Write the expression for $V(t)$ for $t \geq T$.

The series RC circuit is the same as the one we solved for in problem 2 with a voltage source set to 0V. The analysis is otherwise the same, allowing us to use the general solution at the bottom of problem 2c, but shifted to start at time $t = T$ instead:

$$V(t \geq T) = \big(V(T) - V(t \gg T + RC)\big)e^{-\frac{t-T}{RC}} + V(t \gg T + RC)$$

Plugging in for our initial condition and steady state voltage:

$$\boxed{V(t \geq T) = V_0 e^{-\frac{t-T}{RC}}}$$

Stitched together, the voltage trace looks something like the cartoon plotted below. We see it is continuous, as we required.