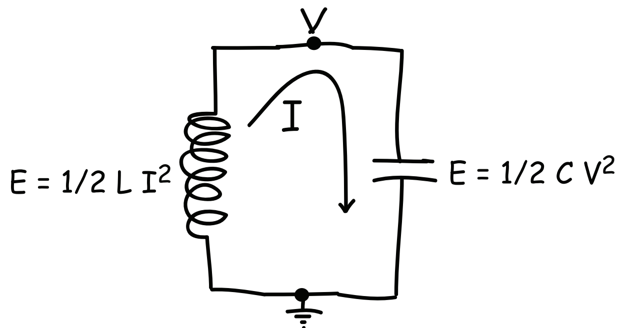
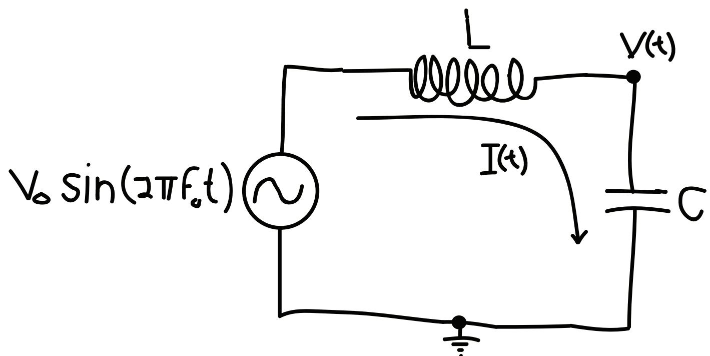
Review of LC Circuits

In a LC circuit, electrical potential energy, stored as the charge across a capacitor, and magnetic potential energy, stored as flux within an inductor, sloshes back and forth in the absence of loss, i.e., with no resistance in the circuit.



Let's consider the dynamics in detail for a sinusoidal drive voltage, $V_{drive} = V_0 \sin(2\pi f_0 t)$.



Recall that the voltage drop across an inductor is:

$$V(t) = L\frac{d\,I(t)}{dt}$$

so that the current through the inductor is just:

$$I(t) = \frac{1}{L}\int_0^t dt'\, V(t')\ .$$

On the other hand, the current flow across a capacitor is:

$$I(t) = C\frac{d\,V(t)}{dt}\,.$$

If we apply Kirchoff's Law for current flow to the node labeled V(t), we get:

$$\frac{1}{L}\int_0^t dt'\left[V(t') - V_0\,\sin(2\pi f_0 t')\right] + C\,\frac{dV(t)}{dt} = 0\,.$$

Thus:

$$\int_0^t dt'\,V(t') + LC\,\frac{dV(t)}{dt} = V_0\,\frac{1}{2\pi f_0}\,\cos(2\pi f_0 t)$$

or, differentiating once:

$$V(t) + LC\,\frac{d^2V(t)}{dt^2} = -V_0\,\sin(2\pi f_0 t)\,.$$

Let's stare at this. One steady-state solution is $V(t) = A\sin(2\pi ft)$, which gives

$$A\sin(2\pi ft) - A\,LC\,(2\pi f)^2\sin(2\pi ft) = -V_0\,\sin(2\pi f_0 t)\,.$$

This is a valid solution so long as:

$$f = f_0$$

and $A - A\,LC\,(2\pi f_0)^2 = -V_0$, or:

$$|A| = V_0\,\frac{1}{1 - LC\,(2\pi f_0)^2}\ .$$

This demonstrates resonance, i.e., the amplitude explodes when $LC = 1/(2\pi f_0)^2$. We identify:

$$\boxed{f_{natural} = \frac{1}{2\pi\sqrt{LC}}}$$

as the natural frequency of this system. Thus the current in the circuit, $I(t) = C\,dV(t)/dt$, is:

$$\boxed{I(t) = -\frac{V_0\ 2\pi f_0\ C}{1 - (f_0/f_{natural})^2}\ \cos(2\pi f_0 t)}.$$

The change in sign, from "-" when $f_0 < f_{natural}$ to "+" when $f_0 > f_{natural}$, corresponds to current leading versus lagging the voltage.